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Note

On finding orientations with the fewest number of vertices with small out-degree

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ABSTRACT

Given an undirected graph, each of the two end-vertices of an edge can "own" the edge. Call a vertex "poor" if it owns at most one edge. We give a polynomial time algorithm for the problem of finding an assignment of owners to the edges which minimizes the number of poor vertices. In the terminology of graph orientation, this means finding an orientation for the edges of a graph which minimizes the number of vertices with out-degree at most 1, and answers a question of Asahiro et al. (2013).

1. Introduction

Let G be a simple[1] undirected graph. An *orientation* of G is a function Λ, which maps each undirected edge $\{u, v\} \in E(G)$ to one of the two possible directed edges (u, v) or (v, u). We let $\Lambda(G)$ be the directed graph whose vertex set is $V(G)$ and whose set of (directed) edges is $\{\Lambda(\{u, v\}) \mid \{u, v\} \in E(G)\}$. For each $u \in V(G)$, the *out-degree of* u *under* Λ is denoted by

$$d^+_\Lambda(u) := \Big|\big\{\{u, v\} \in E(G) \mid \Lambda(\{u, v\}) = (u, v)\big\}\Big|.$$

Fix an integer $k \geq 0$. A vertex $v \in V(G)$ is called Λ-k-*light* (or just k-light, light) if $d^+_\Lambda(v) \leq k$, and if $d^+_\Lambda(v) \geq k$ it is called k-*heavy*. Asahiro et al. [2,1] study the combinatorial optimization problem Min-k-Light which asks for finding an orientation minimizing the number of k-light vertices. For $k = 1$, they exhibit classes of graphs on which the problem can be solved in polynomial time, and they ask the following open question.

Question 1 *([2,1])***.** Is Min-1-Light NP-hard for general graphs?

In this short note, we answer the above question:

Theorem 2. Min-1-Light *on a graph with* n *vertices and* m *edges can be solved by single maximum cardinality matching computation in a graph with* $O(m)$ *vertices and* $O(m^2/n)$ *edges.*

Asahiro et al. [2,1] mention a natural weighted version of the problem: the vertices have costs $c_v \in \mathbb{Q}$, $v \in V(G)$ associated with them, and the objective is to find an orientation which minimizes the expression $\sum_v c_v$ over all orientations Λ, where the sum extends over all 1-light vertices v. Our result also gives the complexity of the weighted case.

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[1] Note that our main reference [1] uses multigraphs, but we can assume w.l.o.g. that graphs are simple.

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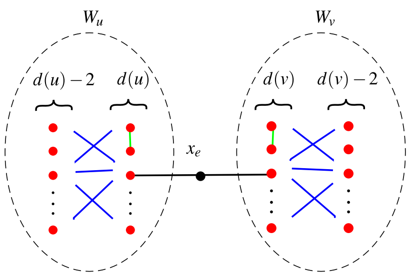

Fig. 1. Two "gadgets" W_u and W_v in the graph G'.

Theorem 3. *For nonnegative weights, weighted* MIN-1-LIGHT *on a graph with* n *vertices and* m *edges can be solved by single maximum weight matching computation in a graph with* $O(m)$ *vertices and* $O(m^2/n)$ *edges.*

If weights can be negative, as a special case (when all weights are -1), it was shown that the problem MAX-1-LIGHT is NP-hard by Asahiro et al. [1].

The proofs of the theorems are in Section 2.

Some notation

We mostly adhere to standard notation. Our (undirected) edges are 2-element subsets of the vertex set. For a vertex $v \in V(G)$, we denote by $\delta(v) := \{e \in E(G) \mid v \in e\}$ the set of all edges incident on v. The degree of a vertex is denoted by $d(v) := |\delta(v)|$.

2. The algorithm for Min-1-Light

We first deal with the case that there are no vertices of degree 1. For such a graph G, construct a graph G' as follows. Start by letting G' be a copy of G. Then replace every edge $e = \{u, v\}$ with a path u, u'_e, x_e, v'_e, v, by adding three new vertices u'_e, x_e, v'_e, and four new edges $\{u, u'_e\}$, $\{u'_e, x_e\}$, $\{x_e, v'_e\}$, $\{v'_e, v\}$. We call the vertices x_e *connecting vertices*, and the edges $\{u'_e, x_e\}$ (and also $\{x_e, v'_e\}$) *connecting edges*, and let $F_u := \{\{u'_e, x_e\} \mid e \in \delta(v)\}$.

Now, for each original vertex v, do the following: replace v with $d(v)-2$ new vertices $v''_1, \dots, v''_{d(v)-2}$. Add $(d(v)-2)\cdot d(v)$ edges between the v''_i and the v'_e, for every i and every $e \in \delta(v)$. Finally, choose two edges $e, f \in \delta(v)$ arbitrarily, and add an edge $g_v := \{v'_e, v'_f\}$, which we call the *special edge.*

In this way, G' contains pairwise disjoint "gadgets" ($\hat{=}$ induced subgraphs) W_v, $v \in V(G)$, each with $d(v)-2+d(v)$ vertices (includes $v''_1, \dots, v''_{d(v)-2}$ and also v'_e, for every $e \in \delta(v)$) and $(d(v)-2)\cdot d(v)+1$ edges. If $\{u, v\} \in E(G)$, then the gadgets W_u and W_v are joined to the connecting vertex $x_{\{u,v\}}$. Cf. Fig. 1. With $n := |V(G)|$ and $m := |E(G)|$, the resulting graph G' has

$$m + \sum_{v \in V(G)} \big(d(v) - 2 + d(v)\big) = 5m - 2n \text{ vertices, and}$$

$$\sum_{v \in V(G)} \big((d(v)-2)d(v) + 1 + d(v)\big) \le \frac{4m^2}{n} + n - 2m \text{ edges}$$

(by using the Cauchy–Schwarz inequality).

The following fact is crucial in the construction.

Lemma 4. *Let* M *be a maximal matching in* G'*. For each* $v \in V(G)$*, there exists a matching* N_v *which satisfies the following conditions:*

1. N_v *differs from* M *only on* $E(W_v)$.
2. $N_v = M$ *or* $|N_v \cap E(W_v)| = |M \cap E(W_v)| + 1$.
3. *With* $k := |M \cap F_v|$*, we have*

$$\big|N_v \cap \big(E(W_v) \cup F_v\big)\big| = \begin{cases} d(v)-1, & \text{if } 0 \le k \le 1 \\ d(v), & \text{if } k \ge 2. \end{cases} \tag{1}$$

Proof. Let M be a maximal matching. If $k = 0$ and the special edge g_v is not in M, then, to obtain N_v, we replace $M \cap E(W_v)$ with the edges of a perfect matching of W_v, which consists of g_v plus a perfect bipartite matching between the v''_i and the v'_e. This increases the number of edges in the matching by 1. If $k \ge 2$ and $g_v \in M$, then at least two of the vertices v''_i, $i = 1, \dots, d(v) - 2$ are exposed. To obtain N_v, we delete g_v from M and add two edges from the exposed vertices in v''_i, $i = 1, \dots, d(v) - 2$, to the end-vertices of g_v, thus increasing the number of W_v-edges in the matching by 1. In all other cases, we leave M unchanged: $N_v := M$.