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Quality bounds for binary tomography with arbitrary projection matrices

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ABSTRACT

Binary tomography deals with the problem of reconstructing a binary image from a set of its projections. The problem of finding binary solutions of underdetermined linear systems is, in general, very difficult and many such solutions may exist. In a previous paper we developed error bounds on differences between solutions of binary tomography problems restricted to projection models where the corresponding matrix has constant column sums. In this paper, we present a series of computable bounds that can be used with any projection model. In fact, the study presented here is not restricted to tomography and works for more general linear systems.

We report the results of computational experiments for some phantom images, focused on parallel and fan beam projection models. Our results show that in some cases the computed bounds can be used to prove that the difference between binary solutions must be small, even if the corresponding linear system is severely underdetermined.

1. Introduction

Binary tomography deals with the problem of reconstructing a binary image from its projections [11]. Projection images of an object are typically recorded using a scanning device, which employs a beam that is transmitted through the object (e.g. photons, electrons). An array of detectors records the beam intensity after the beam–object interaction, resulting in a projection of the object. Due to dose constraints or geometrical constraints on the angles for which projections can be acquired, the set of angles for which projections are acquired is often limited [9,13]. By exploiting the fact that the reconstructed image must be binary, it is often possible to compute useful reconstructions even if just a few projections are available [11]. However, such underdetermined binary tomography problems can have a large number of binary solutions, making it important to have a quality measure for the reconstruction with respect to the unknown original image.

The reconstruction problem can be modeled as a system of linear equations. The matrix that encodes these equations is known as the *projection matrix*. Depending on the particular type of tomography problem, different models can be used to define the projection matrix. In the *grid model*, which is a common model in *discrete tomography* [10,11], an image is formed by assigning a value to each point in a regular grid. In the *line model* and the *strip model*, one models a continuous image that is approximated on a grid of pixels having a constant gray value within each pixel. The results in this paper are restricted to

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binary tomography, but are not restricted to any particular projection model. The approach can be used for the grid model, but also for projection models used in continuous tomography, as will be further detailed in Section 7.1.

Our results are somewhat related to the stability problem in discrete tomography, which has been studied by several authors [1–3,8,15–17] for the grid model. The stability problem deals with the question whether a small perturbation of the observed data results in a small perturbation of the reconstruction. In our work, we deal with the problem how large the difference can be between binary solutions of a reconstruction problem, i.e. binary images that have *identical* projections.

Recently, quite general results were obtained allowing the computation of error bounds between images in binary tomography for any number of projection angles. In [18], a method is presented for computing bounds on the maximum distance between binary solution of tomography problems defined on a discrete grid. Related bounds for more general projection models are derived in [5]. Although in the latter article the tomography problem is presented in a general setting of a linear system of equations, the approach is limited to projection models for which the associated *projection matrix* has constant column sums (i.e. identical sums for all columns). The property of constant column sums holds in particular cases (e.g. the strip model for parallel beam tomography), but limits the application of the results to restricted cases (see Section 7.1 for projection models that do not satisfy this assumption). Although the bounds can still be approximated if this assumption is not completely satisfied, it is then no longer clear if they really provide a quality *guarantee*.

In this article, we derive error bounds that are more general than those in [5], as they can be applied to basically any problem modeled as an underdetermined algebraic linear system of equations. In particular, all relevant models of binary tomography fit within our generalized problem setting. Our generalization of the results from [5] is not at all automatic. New concepts and proofs are introduced to overcome the dependency on constant column sums, paving the way towards practical error bounds for binary tomography, which can be used, for example, when using a cone-beam projection model [20] or in the case of truncated projection data [19].

Although our focus is on tomography, our results are more general. We therefore consider the following general problem, of finding a binary vector that satisfies

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{1}$$

a consistent and underdetermined linear system of algebraic equations with $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{m\times n}$, $m < n$, the vector of unknowns $\boldsymbol{x} = (x_j) \in \mathbb{R}^n$ and the right-hand side $\boldsymbol{b} = (b_i) \in \mathbb{R}^m$.

Finding a binary solution of Eq. (1) is often a very difficult problem and several binary solutions may exist. A given binary solution does not have to be close to another binary solution. In practice, the right hand side vector $\boldsymbol{b}$ is often obtained from an *original* binary vector $\boldsymbol{x}$ by a certain measurement procedure, modeled as the matrix $\boldsymbol{A}$. For a given measurement vector $\boldsymbol{b}$, it is unlikely that all binary solutions are representative solutions of the specific problem which yielded $\boldsymbol{b}$, since some solutions of Eq. (1) may be meaningless for physical problems. In such cases, it can be important to know how different these solutions can be. If one can give a bound on the maximum difference between two solutions, this also bounds the maximum difference between the *ground truth vector* (i.e., from which the vector of measured data $\boldsymbol{b}$ was obtained) and any other solution.

This article is structured as follows. In Section 2, we establish the notation which will be used throughout this article. In Section 3, different versions of bounds on the Euclidean norm of binary solutions are introduced. In Section 4, a general bound is derived on the difference between two binary solutions. Section 5 deals with bounds that are based on properties of the binary vectors that are obtained by rounding the minimum norm solution. These bounds are refined with a different approach in Section 6. Section 7 presents a series of simulation experiments for fan and parallel beam binary tomography and their results. From these results, the practical value of the proposed bounds can be evaluated for different kinds of problems. Section 8 concludes the article.

2. Notation and the minimum norm solution

For a given matrix $\boldsymbol{A}$ and given right-hand side $\boldsymbol{b}$, let $S_{\boldsymbol{A}}(\boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$, the set of all real-valued solutions corresponding with the given data. A *binary vector* corresponds with a vector $\bar{\boldsymbol{x}} \in \{0, 1\}^n$. Let $\bar{S}_{\boldsymbol{A}}(\boldsymbol{b}) = S_{\boldsymbol{A}}(\boldsymbol{b}) \cap \{0, 1\}^n$, the set of *binary solutions* of the system.

Throughout this article, we use the vector $\boldsymbol{0}_t \in \mathbb{R}^t$ (for an integer $t > 0$), to denote a column vector consisting of t 0's, the vector $\boldsymbol{1}_t \in \mathbb{R}^t$ to denote a column vector consisting of t 1's and the identity matrix $\boldsymbol{I}_t \in \mathbb{R}^{t\times t}$. However, we often use the vectors $\boldsymbol{0}$ and $\boldsymbol{1}$ and the identity matrix $\boldsymbol{I}$ without specifying their dimension, as it does not compromise the understanding and clarity of the proofs.

For any two vectors $\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}} \in \{0, 1\}^n$, define the *difference set* $\mathrm{D}(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}) = \{i : \bar{u}_i \neq \bar{v}_i\}$ and the *number of differences* $\mathrm{d}(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}) = \#\mathrm{D}(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}})$, where the symbol # denotes the cardinality operator for a finite set. Note that $\mathrm{d}(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}) = \|\bar{\boldsymbol{u}} - \bar{\boldsymbol{v}}\|_1$.

For the following sections, consider the problem of finding a binary solution of a fixed linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ called the *binary solution problem*.

As the matrix $\boldsymbol{A}$ is not a square matrix, and may not have full rank, it does not have an inverse. Recall that the *Moore–Penrose pseudo-inverse* of an $m \times n$ matrix $\boldsymbol{A}$ is an $n \times m$ matrix $\boldsymbol{A}^\dagger$, which can be uniquely characterized by the two geometric conditions

$$\boldsymbol{A}^\dagger\boldsymbol{b} \perp \mathcal{N}(\boldsymbol{A}) \quad \text{and} \quad (\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^\dagger)\boldsymbol{b} \perp \mathcal{C}(\boldsymbol{A}), \quad \forall \boldsymbol{b} \in \mathbb{R}^m,$$

where $\mathcal{N}(\boldsymbol{A})$ is the nullspace of $\boldsymbol{A}$ and $\mathcal{C}(\boldsymbol{A})$ is the *column space* of $\boldsymbol{A}$ [7, p. 15].