Discrete Applied Mathematics 180 (2015) 11–24

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Excluding cycles with a fixed number of chords

Pierre Aboulker [a,*], Nicolas Bousquet [b]

[a] *Concordia University, Montréal, Québec, Canada*

[b] *LIRMM, Université Montpellier II, France*

ARTICLE INFO

Article history:
Received 5 April 2013
Received in revised form 10 February 2014
Accepted 4 August 2014
Available online 4 September 2014

Keywords:
Chords
Chromatic number
Chi-boundedness

ABSTRACT

Trotignon and Vušković completely characterized graphs that do not contain cycles with exactly one chord. In particular, they show that such a graph G has chromatic number at most $\max(3, \omega(G))$. We generalize this result to the class of graphs that do not contain cycles with exactly two chords and the class of graphs that do not contain cycles with exactly three chords.

More precisely we prove that graphs with no cycle with exactly two chords have chromatic number at most 6. And a graph G with no cycle with exactly three chords has chromatic number at most $\max(96, \omega(G) + 1)$.

1. Introduction

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors needed to vertex-color G such that every two adjacent vertices receive distinct colors. A clique is a graph such that every two vertices are adjacent. The *clique number* of a graph G, denoted by $\omega(G)$, is the number of vertices of the largest clique in G. A class of graphs is *hereditary* if, for any graph G in the class, every subgraph of G is in the class. We say that a graph G is H*-free* if G does not contain the graph H as an induced subgraph. If $\mathcal{H}$ is a class of graphs we say that a graph G is $\mathcal{H}$-free if for any $H \in \mathcal{H}$, G is H-free. Classes of graphs defined by forbidding some graphs as induced subgraphs are clearly hereditary.

It is clear that $\omega(G)$ is a lower bound of $\chi(G)$ since vertices of a clique are colored with pairwise distinct colors. Gyárfás introduced the following notion [7]: a class of graphs $\mathcal{C}$ is χ*-bounded* if there exists a function f such that for every graph $G \in \mathcal{C}$ and every subgraph H of G, $\chi(H) \leq f(\omega(H))$. Observe that, in order to prove that a hereditary class $\mathcal{C}$ is χ-bounded by a function f, it is enough to prove that for any graph G in $\mathcal{C}$, $\chi(G) \leq f(\omega(G))$. For instance, the class of graphs χ-bounded by the function $f(x) = x$ are known as *perfect graphs*. Chudnovsky, Robertson, Seymour, and Thomas proved [4] that perfect graphs are exactly the graphs that neither have odd cycles of length at least 5 nor complements of odd cycles of length at least 5 as induced subgraphs, solving the famous strong perfect graph conjecture proposed by Berge [2]. So, a natural question arises:

Question 1.1. What kind of induced structure is needed to be forbidden in order to get a χ-bounded class?

Let us now survey some results on χ-boundedness by emphasizing what different meanings "structure" can take.

If H is a graph, we denote by *Forb(H)* the class of H-free graphs. A first way to tackle the problem is to determine for which graphs H, Forb(H) is χ-bounded. For example, it is proved in [7] that Forb(P_k) is χ-bounded (where P_k denotes the chordless path of length k). In [6], Erdős proved that there exist graphs with arbitrarily large chromatic number and arbitrarily large girth. So, if H contains a cycle, Forb(H) is not χ-bounded (since a graph of girth k does not contain a cycle of length at most

* Corresponding author. Tel.: +33 679297478; fax: +33 157279409.
E-mail addresses: pierreaboulker@gmail.com (P. Aboulker), bousquet@lirmm.fr (N. Bousquet).

http://dx.doi.org/10.1016/j.dam.2014.08.006
0166-218X/© 2014 Published by Elsevier B.V.

$k-1$). It is actually conjectured in [7] that Forb(H) is χ-bounded if and only if H is a forest. The deeper results concerning this conjecture are certainly results of Kierstead and Penrice [8] and Kierstead and Zhu [9] proving that the conjecture holds for every tree of radius at most 2 and several trees of radius 3. For making further progress, we need to forbid a class of trees $\mathcal{H}$ such that $\mathcal{H}$ contains trees with arbitrarily large diameter.

A second way to forbid induced structures is the following: fix a graph H, and forbid every induced subdivision of H. We denote by *Forb*(H)* the class of graphs that do not contain induced subdivisions of H. The class Forb*(H) has been proved to be χ-bounded for a number of examples. The most beautiful one is certainly the proof by Scott [11] that for any forest F, Forb*(F) is χ-bounded. In the same paper he conjectured that, for any graph H, Forb*(H) is χ-bounded. Unfortunately, this conjecture has recently been disproved by Kozik et al. [10]. Based on this work, Chalopin et al. [3] gave a description of a number of graphs H for which Forb*(H) is not χ-bounded. There is no general conjecture on which H has to be forbidden in order to ensure Forb*(H) is χ-bounded.

A third way is to forbid a graph H for which some edges can be subdivided but some cannot. More generally, to forbid a class of graphs $\mathcal{H}$ such that, for each $H \in \mathcal{H}$, some edges can be subdivided and some cannot. A few classes defined in this way have been studied (see [1] and [12] for instance). In [12], Trotignon and Vušković proved that the class of graphs that do not contain cycles with a unique chord is χ-bounded by the function $\max(3, \omega(G))$. Forbidding cycles with a unique chord is equivalent to forbidding a diamond (a diamond is a cycle of length 4 with a diagonal) such that all edges but the diagonal can be subdivided.

A *k-cycle* is a cycle with exactly k chords. We call $\mathcal{C}_k$ the class of (k-cycle)-free graphs i.e. the class of graphs that do not contain cycles with exactly k chords. So, the result of Trotignon and Vušković may be rephrased as follows: $\mathcal{C}_1$ is χ-bounded.

The two main results of this paper are that both $\mathcal{C}_2$ (see Theorem 4.1) and $\mathcal{C}_3$ (see Theorem 5.3) are χ-bounded. The statement of Theorem 4.1 deals with a super-class of $\mathcal{C}_2$, see Section 4 for more details. Since graphs which do not contain 2-cycles do not contain K_4 as an induced subgraph, proving χ-boundedness is equivalent to proving that the chromatic number is bounded. We actually prove that the chromatic number in this case is at most 6. An immediate lower bound on the chromatic number is 3, given by odd cycles. Close the gap between these two values is an interesting question.

The class $\mathcal{C}_3$, in contrast to $\mathcal{C}_2$ that does not admit graphs with cliques larger than the triangle (because K_4 is a 2-cycle), admits graphs containing arbitrarily large cliques. We prove that the chromatic number of a graph $G \in \mathcal{C}_3$ is at most $\max(96, \omega(G)+1)$. In addition we provide examples of graphs G with arbitrarily large clique such that $\chi(G) = \omega(G)+1$, showing that our bound is asymptotically tight. Nevertheless, the lower bound of 96 is surely far away from an optimal bound for graphs in $\mathcal{C}_3$ that do not contain large cliques.

Here is an outline of the paper. In Section 2, we give some terminology and in Section 3 we describe the general method used in the proofs. Section 4 is concerned with the class $\mathcal{C}_2$ and Section 5 is concerned with the class $\mathcal{C}_3$. We also propose the following conjecture suggested by our results:

Conjecture 1.2. *For any integer $k \geq 4$, $\mathcal{C}_k$ is χ-bounded.*

2. Terminologies and notations

For standard definitions on graphs, the reader should refer to classical books of graph theory, such as [5]. Let G be a graph, x a vertex of G and S a subset of vertices of G. We denote by $N(x)$ the set of neighbors of x, by $N_S(x)$ the set of neighbors of x in S, and by $N(S)$ the set of vertices of $V(G) \setminus S$ that has a neighbor in S. We denote by $d(x)$ the degree of x and by $d_S(x)$ the degree of x in S, i.e. the number of neighbors of x in S. We denote by $G[S]$ the subgraph of G induced by S, and $G \setminus S$ denotes $G[V(G) \setminus S]$. A set S of vertices is a *cutset* of G if $G \setminus S$ has more than one connected component. If S induces a clique, then S is a *clique cutset*. If $\{x\}$ is a cutset of G, then x is a *cutvertex*. Note that a cutvertex is a clique cutset.

A *path P* is a sequence of distinct vertices $p_1p_2\dots p_k$ with $k \geq 1$, such that p_ip_{i+1} is an edge for every $1 \leq i < k$. For every $1 \leq i < k$, the edge p_ip_{i+1} is an *edge of P*. Vertices p_1 and p_k are the *endpoints* of P, and $p_2 \dots p_{k-1}$ is the *interior* of P. P is referred to as a *p_1p_k-path*. For $1 \leq i \leq j \leq k$, we write $p_iPp_j := p_i \dots p_j$, $\mathring{P} := p_2 \dots p_{k-1}$, $\mathring{p}_iP\mathring{p}_j := p_{i+1} \dots p_{j-1}$.

A cycle C is a sequence of vertices $p_1p_2 \dots p_kp_1$, $k \geq 3$, such that $p_1 \dots p_k$ is a path and p_1p_k is an edge. Edges p_ip_{i+1}, for $1 \leq i < k$, and edge p_1p_k are called the *edges of C*. Let Q be a path or a cycle. The *length* of Q is its number of edges. An edge $e = uv$ of G is a *chord* of Q if $u, v \in V(Q)$, but uv is not an edge of Q. A path or a cycle Q in a graph G is *chordless* if no edge of G is a chord of Q.

A graph G is *complete k-partite* if $V(G)$ can be partitioned into k non-empty subsets $A_1, \dots, A_k$ such that, for $i = 1, \dots, k$, A_i is a stable set and, for any $\{i, j\} \subseteq \{1, \dots, k\}$, G contains all possible edges between A_i and A_j. Sets A_i are called the *partition sets* of G. The graph G is denoted by $K_{a_1,\dots,a_k}$ where $a_i = |A_i|$ for $i = 1, \dots, k$. If $k = 2$, then G is said to be a *complete bipartite graph* and if $k = 3$, G is said to be a *complete tripartite graph*. The graph $K_{1,1,2}$ is called a *diamond*.

3. Preliminaries

We mentioned that $\mathcal{C}_1$ was already proved to be χ-bounded. We use this result for graphs in $\mathcal{C}_1$ that contain no K_4. Formally, Trotignon and Vušković proved the following:

Theorem 3.1 (*Trotignon and Vušković [12]*)**.** *If $G \in \mathcal{C}_1$ and $\omega(G) \leq 3$, then $\chi(G) \leq 3$.*