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On the tractability of some natural packing, covering and partitioning problems

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ABSTRACT

In this paper we fix 7 types of undirected graphs: paths, paths with prescribed endvertices, circuits, forests, spanning trees, (not necessarily spanning) trees and cuts. Given an undirected graph $G = (V, E)$ and two "object types" A and B chosen from the alternatives above, we consider the following questions. **Packing problem:** can we find an object of type A and one of type B in the edge set E of G, so that they are edge-disjoint? **Partitioning problem:** can we partition E into an object of type A and one of type B? **Covering problem:** can we cover E with an object of type A, and an object of type B? This framework includes 44 natural graph theoretic questions. Some of these problems were well-known before, for example covering the edge-set of a graph with two spanning trees, or finding an s-t path P and an s'-t' path P' that are edge-disjoint. However, many others were not, for example can we find an s-t path $P \subseteq E$ and a spanning tree $T \subseteq E$ that are edge-disjoint? Most of these previously unknown problems turned out to be NP-complete, many of them even in planar graphs. This paper determines the status of these 44 problems. For the NP-complete problems we also investigate the planar version, for the polynomial problems we consider the matroidal generalization (wherever this makes sense).

1. Introduction

In this paper we consider undirected graphs. The node set of a graph $G = (V, E)$ is sometimes also denoted by $V(G)$, and similarly, the edge set is sometimes denoted by $E(G)$. A **subgraph** of a graph $G = (V, E)$ is a pair (V', E') where $V' \subseteq V$ and $E' \subseteq E \cap (V' \times V')$. A graph is called **subcubic** if every node is incident to at most 3 edges, and it is called **subquartic** if every node is incident to at most 4 edges. By a **cut** in a graph we mean the set of edges leaving a nonempty proper subset V' of the nodes (note that we do not require that V' and $V - V'$ induce a connected graph). We use standard terminology and refer the reader to [9] for what is not defined here.

We consider 3 types of decision problems with 7 types of objects. The three types of problems are: packing, covering and partitioning, and the seven types of objects are the following: paths (denoted by a P), paths with specified endvertices (denoted by P_{st}, where s and t are the prescribed endvertices), (simple) circuits (denoted by C: by that we mean a closed walk of length at least 2, without edge- and node-repetition), forests (F), spanning trees (SpT), (not necessarily spanning) trees (T), and cuts (denoted by Cut). Let $G = (V, E)$ be a **connected** undirected graph (we assume connectedness in order

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Table 1: 25 partitioning problems.

Problem	Status	Reference	Remark
P + P	**NPC**	Theorem 5	**NPC** for subquartic planar
P + P_{st}	**NPC**	Theorem 5	**NPC** for subquartic planar
P + C	**NPC**	Theorem 5	**NPC** for subquartic planar
P + T	**NPC**	Theorem 5	**NPC** for subquartic planar
P + SpT	**NPC**	Theorem 5	**NPC** for subquartic planar
P + F	**NPC**	Theorem 3 (and Theorem 5)	**NPC** for subcubic planar
P_{st} + $P_{s't'}$	**NPC**	Theorem 5	**NPC** for subquartic planar
P_{st} + C	**NPC**	Theorem 5	**NPC** for subquartic planar
P_{st} + T	**NPC**	Theorem 5	**NPC** for subquartic planar
P_{st} + SpT	**NPC**	Theorem 5	**NPC** for subquartic planar
P_{st} + F	**NPC**	Theorem 3 (and Theorem 5)	**NPC** for subcubic planar
C + C	**NPC**	Theorem 5	**NPC** for subquartic planar
C + T	**NPC**	Theorem 5	**NPC** for subquartic planar
C + SpT	**NPC**	Theorem 5	**NPC** for subquartic planar
C + F	**NPC**	Theorem 3 (and Theorem 5)	**NPC** for subcubic planar
T + T	**NPC**	Pálvölgyi [19]	planar graphs?
T + SpT	**NPC**	Theorem 6	planar graphs?
F + F	**P**	Kishi and Kajitani [14], Kameda and Toida [12] (Nash-Williams [18])	in **P** for matroids: Edmonds [7]
SpT+SpT	**P**	Kishi and Kajitani [14], Kameda and Toida [12], (Nash-Williams [17], Tutte [24])	in **P** for matroids: Edmonds [7]
Cut + Cut	**P**	if and only if bipartite (and $\|V\| \geq 3$)	
Cut + F	**NPC**	Theorem 7	planar graphs?
Cut + C	**NPC**	Theorem 3	**NPC** for subcubic planar
Cut + T	**NPC**	Theorem 3	**NPC** for subcubic planar
Cut + P	**NPC**	Theorem 3	**NPC** for subcubic planar
Cut + P_{st}	**NPC**	Theorem 3	**NPC** for subcubic planar

to avoid trivial case-checkings) and A and B two (not necessarily different) object types from the 7 possibilities above. The general questions we ask are the following:

- **Packing problem** (denoted by A $\wedge$ B): can we **find two edge-disjoint subgraphs** in G, one of type A and the other of type B?
- **Covering problem** (denoted by A $\cup$ B): can we **cover the edge set** of G with an object of type A and an object of type B?
- **Partitioning problem** (denoted by A + B): can we **partition the edge set** of G into an object of type A and an object of type B?

Let us give one example of each type. A typical partitioning problem is the following: decide whether the edge set of G can be partitioned into a spanning tree and a forest. Using our notations this is Problem SpT + F. This problem is in **NP** $\cap$ **co-NP** by the results of Nash-Williams [18], polynomial algorithms for deciding the problem were given by Kishi and Kajitani [14], and Kameda and Toida [12].

A typical packing problem is the following: given four (not necessarily distinct) vertices $s, t, s', t' \in V$, decide whether there exists an s–t path P and an s'–t'-path P' in G, such that P and P' do not share any edge. With our notations this is Problem $P_{st} \wedge P_{s't'}$. This problem is still solvable in polynomial time, as was shown by Thomassen [23] and Seymour [22].

A typical covering problem is the following: decide whether the edge set of G can be covered by a path and a circuit. In our notations this is Problem $P \cup C$. Interestingly we found that this simple-looking problem is NP-complete.

Let us introduce the following short formulation for the partitioning and covering problems. If the edge set of a graph G can be partitioned into a type A subgraph and a type B subgraph, then we will also say that **the edge set of** G **is** $A + B$. Similarly, if there is a solution of Problem $A \cup B$ for a graph G, then we say that **the edge set of** G **is** $A \cup B$.

The setting outlined above gives us 84 problems. Note however that some of these can be omitted. For example P $\wedge$ A is trivial for each possible type A in question, because P may consist of only one vertex. By the same reason, T $\wedge$ A and F $\wedge$ A type problems are also trivial. Furthermore, observe that the edge-set $E(G)$ of a graph G is F + A $\Leftrightarrow$ $E(G)$ is F $\cup$ A $\Leftrightarrow$ $E(G)$ is T $\cup$ A $\Leftrightarrow$ $E(G)$ is SpT $\cup$ A: therefore we will only consider the problems of form F + A among these for any A. Similarly, the edge set $E(G)$ is F + F $\Leftrightarrow$ $E(G)$ is T + F $\Leftrightarrow$ $E(G)$ is SpT + F: again we choose to deal with F + F. We can also omit the problems Cut + SpT and Cut $\wedge$ SpT because a cut and a spanning tree can never be disjoint.

The careful calculation gives that we are left with 44 problems. We have investigated the status of these. Interestingly, many of these problems turn out to be NP-complete. Our results are summarized in Tables 1–3. We note that in our NP-completeness proofs we always show that the considered problem is NP-complete even if the input graph is simple. On the other hand, the polynomial algorithms given here always work also for multigraphs (we allow parallel edges, but we forbid loops). Some of the results shown in the tables were already proved in the preliminary version [5] of this paper: namely we have already shown the NP-completeness of Problems P + T, P + SpT, P_{st} + T, P_{st} + SpT, C + T, C + SpT, T + SpT, P_{st} $\wedge$ SpT, and C $\wedge$ SpT there.

Problems P_{st} + SpT and T + SpT were posed in the open problem portal called "EGRES Open" [8] of the Egerváry Research Group. Most of the NP-complete problems remain NP-complete for planar graphs, though we do not know yet the status of Problems T + T, T + SpT, Cut + F, P_{st} $\wedge$ SpT, and C $\wedge$ SpT for planar graphs, as indicated in the table.