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The height and width of bargraphs

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ABSTRACT

A bargraph is a lattice path in $\mathbb{N}_0^2$ with three allowed steps: the up step $u = (0, 1)$, the down step $d = (0, -1)$ and the horizontal step $h = (1, 0)$. It starts at the origin with an up step and terminates as soon as it intersects the x-axis again. A down step cannot follow an up step and vice versa. The *height* of a bargraph is the maximum y coordinate attained by the graph. The width is the horizontal distance from the origin till the end. For bargraphs of fixed semi-perimeter n we find the generating functions for the total height and the total width and hence find asymptotic estimates for the average height and the average width. Our methodology makes use of a bijection between bargraphs and *uudd*-avoiding Dyck paths.

1. Introduction

A bargraph is a lattice path in $\mathbb{N}_0^2$, analogous to Dyck or Motzkin paths. There are three allowed steps: the up step $(0, 1)$, the down step $(0, -1)$ and the horizontal step $(1, 0)$ denoted by u, d and h respectively. The bargraph starts at the origin with an up step and terminates as soon as the path intersects the x-axis again. A down step cannot follow an up step and vice versa. The *height* of a bar graph is the maximum y coordinate attained by the graph, the *width* is the maximum x coordinate and the semi-perimeter is the sum of the number up and horizontal steps.

So for example, we have the bargraph (see Fig. 1).

Bargraphs have been studied particularly in statistical physics; see [4–6,9–14]. Other names used for bargraphs are wall polyominoes [7] or skylines [9]. In [1–3], the first three authors investigate various combinatorial statistics associated with bargraphs.

In this paper, we find the generating function for bargraphs of height at most h and use this to find an asymptotic expression for the average height of bargraphs of semi-perimeter n. We also consider the width or horizontal semi-perimeter of bargraphs with fixed total semi-perimeter.

A main tool for studying statistics of interest is a decomposition of bargraphs which is based on the first return to level one; see [13]. This was also used by Bousquet-Mélou and Rechnitzer in [5], where they called it the *wasp-waist decomposition*. The current authors have also made extensive use of it in [1–3].

The generating function for all bargraphs can be found in [5] amongst others. It is given by

$$B(x, y) = \frac{1 - x - y - xy - \sqrt{(1 - x - y - xy)^2 - 4x^2y}}{2x} \tag{1.1}$$

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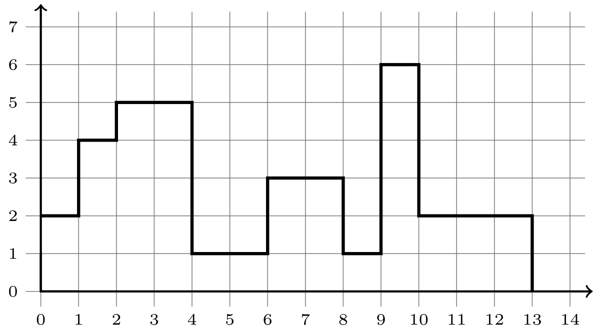

Fig. 1. A bargraph of height 6, width 13 and semi-perimeter 25.

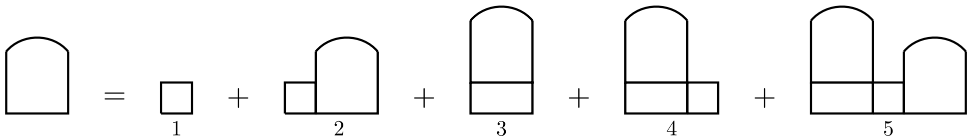

Fig. 2. Wasp-waist factorisation of bargraphs.

where x counts the number of horizontal steps and y counts the number of up steps. If we substitute $z = y = x$ we obtain the generating function for the semi-perimeter counted by z, often called the isotropic generating function

$$B(z,z) = \frac{1-2z-z^2-\sqrt{1-4z+2z^2+z^4}}{2z}. \tag{1.2}$$

To find the asymptotics for $B(z,z)$, we must first compute the dominant singularity ρ which is the positive root of $D := 1-4z+2z^2+z^4 = 0$. We find

$$\rho = \frac{1}{3}\left(-1 - \frac{4\times 2^{2/3}}{(13+3\sqrt{33})^{1/3}} + \left(2(13+3\sqrt{33})\right)^{1/3}\right) = 0.295598\ldots. \tag{1.3}$$

Then $B(z,z) \sim \psi_1(z)(1-z/\rho)^{1/2}$ and by singularity analysis (see [8]) we have

$$[z^n]B(z,z) \sim \frac{\psi_1(\rho)\rho^{-n}}{2\sqrt{\pi n^3}} \tag{1.4}$$

where

$$\psi_1(\rho) = \sqrt{\frac{1-\rho-\rho^3}{\rho}}.$$

2. The generating function for bargraphs of height at most *h*

For a fixed $h \geq 1$, let $G(x,y,h)$ be the generating function for bargraphs in which x marks the total number of horizontal steps, y marks the total number of ascent steps and for which the height of the graph is less or equal to h.

For simplicity we will write $G_h := G(x,y,h)$.

Following Bousquet-Mélou and Rechnitzer in [5], and the authors own use in [1–3] and in [13], the "wasp-waist" decomposition for G_h is represented symbolically in Fig. 2.

Restricting all heights to at most h, the decomposition yields

$$G_h = \underbrace{xy}_{1} + \underbrace{xG_h}_{2} + \underbrace{yG_{h-1}}_{3} + \underbrace{xyG_{h-1}}_{4} + \underbrace{xG_{h-1}G_h}_{5}.$$

Solving this for G_h, we obtain the continued fraction type recursion

$$G_h = \frac{xy + (1+x)yG_{h-1}}{1-x-xG_{h-1}}. \tag{2.1}$$

In order to solve this recursion, we write the rational function G_h as $\frac{p(h)}{q(h)}$, and so obtain from (2.1)

$$\frac{p(h)}{q(h)} = \frac{xyq(h-1) + (1+x)yp(h-1)}{(1-x)q(h-1) - xp(h-1)}. \tag{2.2}$$