Discrete Applied Mathematics 180 (2015) 81–88

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On graphs with excess or defect 2

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ARTICLE INFO

Article history:
Received 2 March 2014
Received in revised form 28 June 2014
Accepted 21 July 2014
Available online 7 August 2014

Keywords:
Moore bound
Excess
Cage number
Defect
Degree/diameter problem

ABSTRACT

We prove that a graph with odd minimal degree, girth 9 and excess 2, i.e. a graph with two more vertices than the Moore bound, cannot exist. In this proof we discover a link to certain elliptic curves. Furthermore, we prove the non-existence of graphs with excess 2 for girth higher than 9 and various valencies under certain congruence conditions. The results can be modified to fit the degree/diameter problem and lead to similar statements for graphs with defect 2. Amongst others we will prove that there is no graph with odd maximal degree, diameter 4 and defect 2. Together with previous results about the degree/diameter problem this completes the case of diameter 4 and defect 2.

1. Introduction

The Moore bound $m(d,k) = 1 + d\sum_{i=0}^{k-1}(d-1)^i$ is a lower bound for the number of vertices of a graph with girth $g = 2k+1$ and minimal degree d, but it is also an upper bound for the number of vertices of a graph with diameter $D = k$ and maximal degree d. Graphs which equal this bound are called Moore graphs. Trivial examples are the cycle C_g and the clique K_{d+1}. Hoffmann and Singleton [13], Bannai and Ito [1] and Damerell [7] showed that non-trivial graphs tight to this bound can only exist for girth 5 (respectively diameter 2) and degree 3, 7, 57. In [13] it is shown that the Petersen graph and the Hoffmann–Singleton graph are the unique examples with $d \in \{3, 7\}$. The case of $d = 57$ is still unsolved. For a graph with girth $g = 2k+1$ the difference of the number of its vertices to the Moore bound $m(d,k)$ is called the excess of the graph. Analogously for a graph with diameter D the difference of the Moore bound $m(d,D)$ to the number of its vertices is called the defect of the graph. Brown showed in [5] that there are no graphs with girth 5 and excess 1 and Erdős, Fajtlowicz and Hoffmann showed in [9] that there are none with diameter 2 and defect 1 except the cycle C_4. Finally, Bannai and Ito showed in [2] that for $d \geq 3$ there are no graphs with excess 1 and girth $g \geq 7$ or defect 1 and diameter $D \geq 3$. Thus there are no graphs with excess 1 and the cycles C_{2D} are the only graphs with defect 1. A graph with minimal degree d, girth g and excess ϵ is called a graph of type (d, g, ϵ). A graph with maximal degree d, diameter D and defect δ is called a graph of type $(d, D, -\delta)$. It is well known that for small excess or defect (and particularly in our case) such graphs are regular. For example this fact is mentioned in [8].

Proposition 1.1. (i) *Let be $k \geq 2$ and $\epsilon < \sum_{i=0}^{k-1}(d-1)^i$. Then a $(d, 2k+1, \epsilon)$-graph is regular.*
(ii) *Let be $D \geq 2$ and $\delta < \sum_{i=0}^{k-1}(d-1)^i$. Then a $(d, 2k+1, -\delta)$-graph is regular.*

1.1. Known results for graphs with excess or defect 2

There is little known about graphs with excess 2. Four examples can be found in [10,16,20], there are two graphs of type $(3, 5, 2)$, one of $(4, 5, 2)$ and one of $(3, 7, 2)$. The only result for infinitely many values is due to Kovács [15] for girth 5. It states

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http://dx.doi.org/10.1016/j.dam.2014.07.022
0166-218X/

that for every $d \in \mathbb{N}$ odd, which is not of the form $d = l^2 + l + 3$ or $d = l^2 + l - 1$, $l \in \mathbb{Z}_{\geq 0}$, there is no graph of type $(d, 5, 2)$. There are more results known about graphs with defect 2. The five known graphs with defect 2 are summarized in [17] and are of the types $(3, 2, -2)$ (twice), $(4, 2, -2)$, $(5, 2, -2)$ and $(3, 3, -2)$. The non-existence results are summarized in Table 1.

1.2. New results for graphs with excess or defect 2

In our first main result we extend the idea of Kovács and prove that there are no graphs with odd degree d, girth 9 and excess 2.

Theorem 1.2. *Let $d \in \mathbb{N}$ be odd. Then there is no graph of type $(d, 9, 2)$.*

Furthermore, we also prove the non-existence of graphs with excess 2, various odd values of d and girth higher than 9. All these new results are stated in Table 2. We prove analogous statements for defect 2, the main result being that there are no graphs with odd degree d, diameter 4 and defect 2.

Theorem 1.3. *Let $d \in \mathbb{N}$ be odd. Then there is no graph of type $(d, 4, -2)$.*

This solves a case, which was left open in [12]. The authors proved that there is no graph with even d, diameter 4 and defect 2. Thus combined with our result diameter 4 is the first case, which is completely settled and there are no graphs with diameter 4 and defect 2.

Corollary 1.4. *There is no graph with diameter 4 and defect 2.*

2. Main section

2.1. Ingredients from algebra and graph theory

We will prove the Theorems by some spectral considerations on the adjacency matrix. Therefore, we will use some algebraic facts. We will observe in Proposition 2.5 that a graph with excess 2 is closely related to the eigenvalues of the cycle, whose minimal polynomials can be calculated by a certainly known recursion.

Proposition 2.1. *The characteristic polynomial of the cycle of even length is given by $C_{2k}(x) = (x-2)(x+2)E_k(x)^2$, where $E_k(x)$ is defined by*

$$E_0(x) = 0,$$
$$E_1(x) = 1,$$
$$E_i(x) = xC_{i-1}(x) - C_{i-2}(x), \quad i \geq 2.$$

The characteristic polynomial of the cycle of odd length is given by $C_{2k+1}(x) = (x-2)O_k(x)^2$, where $O_k(x)$ is defined by

$$O_0(x) = 1,$$
$$O_1(x) = x + 1,$$
$$O_i(x) = xO_{i-1}(x) - O_{i-2}(x), \quad i \geq 2.$$

This formula is a special case for $d = 2$ of the recursion formula for the characteristic polynomial of Moore graphs [13] and respectively generalized polygon graphs [11]. Furthermore, we will use the following well known fact from linear algebra, a proof can be found in [15].

Lemma 2.2. *Let A be a rational symmetric matrix, $\chi_A(x) \in \mathbb{Q}[x]$ be its characteristic polynomial and $p(x) \in \mathbb{Q}[x]$ an irreducible polynomial with $p(x) | \chi_A(x)$. If $\lambda, \mu \in \mathbb{R}$ are roots of $p(x)$, then the multiplicities $m_A(\lambda)$ and $m_A(\mu)$ of λ and μ as eigenvalues of A fulfill*

$$m_A(\lambda) = m_A(\mu).$$

The last ingredient needed for our main result is the following generalization of Eisenstein's criterion. Although it is a straightforward generalization of the classical statement, for completeness we include a short proof.

Lemma 2.3. *Let $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ be a polynomial of degree n. Furthermore, let p be a prime number with $p | a_i$ for all $0 \leq i < m$, $p \nmid a_m$ and $p^2 \nmid a_0$. Then f has an irreducible factor of degree at least m.*

Proof. Let $g = \sum_{i=0}^{r} b_i x^i$, $h = \sum_{i=0}^{s} c_i x^i$ be two polynomials of degree r respectively s such that $g \cdot h = f$. Then $p | a_0 = b_0 c_0$ and $p^2 \nmid b_0 c_0$. Therefore we can assume without loss of generality that $p | b_0$, but $p \nmid c_0$. Let t be the maximal value such that $p | b_i$ for all $0 \leq i \leq t$. Then $t < r$, otherwise every coefficient of f would be divisible by p. Furthermore,

$$a_{t+1} = b_{t+1} c_0 + \cdots + b_0 c_{t+1} \equiv b_{t+1} c_0 \not\equiv 0 \bmod p.$$

Hence $a_{t+1} = a_m$ and $r \geq t + 1 = m$. If g is reducible, recursive use of this argument on the factors of g proves the assertion. □