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# Some efficiently solvable problems over integer partition polytopes

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ABSTRACT

The integer partition polytope $P_n$ is the convex hull of all integer partitions of $n$. We provide a novel extended formulation of $P_n$, and use it to show that the extremality, adjacency, and separation problems over $P_n$ can be solved by linear programming without the ellipsoid method.



## 1. Introduction

An *integer partition* of $n$ is a nonincreasing sequence of positive integers summing up to $n$. Integer partitions play an important role in a variety of areas of mathematics, in statistical mechanics and theoretical physics, see [1,14] and references therein.

We can and will identify here each partition of $n$ with a nonnegative integer vector $x \in \mathbb{Z}_+^n$, where $x_k$ counts the number of times $k$ appears in the sum. For instance, the partition $1 + 1 + 3 = 5$ corresponds to the vector $x = (2, 0, 1, 0, 0)$. Let $T_n := \{x \in \mathbb{Z}_+^n : \sum_{k=1}^n kx_k = n\}$ be the set of integer partitions of $n$. The *integer partition polytope* is defined to be $P_n := \mathrm{conv}(T_n)$, the convex hull of all integer partitions of $n$.

The polyhedral approach in the integer partition theory gives rise to many appealing questions. Introducing the polytope of integer partitions revealed the previously unknown geometric structure of the set of partitions of any integer. It demonstrated existence of new classes of partitions, in particular *extreme integer partitions*, which are the vertices of the partition polytopes. Each partition of $n$ is a convex combination of some vertices of $P_n$, thus the set of vertices of $P_n$ forms a basis for the set of partitions of $n$. This engenders a special interest in vertices.

As for every polytope, the other key elements of $P_n$ are its facets. They have been studied in [11] and the vertices were studied in [12,13]. However, no combinatorial characterizations of vertices or facets of $P_n$ are available as yet.

In this regard, the following question arises: what is the computational complexity of the problem

- Extremality: for integer partition $x \in T_n$, decide if it is extreme on $P_n$.

It is easy to see that this problem is in co-NP: if $x$ is *not* extreme then, by Caratheodory's theorem, one can exhibit $2 \le r \le n+1$ affinely independent integer partitions $x^1, \ldots, x^r \in T_n$ such that in the unique solution to $\sum \lambda_i x_i = x,\ \sum \lambda_i x_i = 1$, all

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$\lambda_i$ are positive. However, the number of all partitions is exponential in $n$ (see [1]), and therefore finding such $x_i$ by exhaustive search takes exponential time.

The main purpose of this paper is to prove that the extremality problem can be decided in polynomial time with the use of linear programming techniques. The lift and project method that we use to solve this problem occurred to be powerful enough to prove polynomial decidability by linear programming of three more problems, which are canonical in combinatorial optimization. These are

- Adjacency: for extreme partitions $x, y \in T_n$, decide if they are adjacent on $P_n$.
- Separation: for $x \in \mathbb{R}^n$, find $h \in \mathbb{R}^n$ with $h^\intercal x > h^\intercal y \forall y \in P_n$ or assert $x \in P_n$.
- Optimization: for $c \in \mathbb{R}^n$, find $x^* \in P_n$ which attains $\min\{c^\intercal x : x \in P_n\}$.

Since $P_n$ is a knapsack type polytope, the optimization problem over $P_n$ can be solved in polynomial time by dynamic programming, see, for example, [6]. This implies that all other problems listed above can also be solved in polynomial time by multiple applications of the ellipsoid method [4]. However, the ellipsoid method is a very heavy tool with a very large running time. In this paper we show that all problems can in fact be solved in polynomial time using linear programming *without* the ellipsoid method. We prove the following theorem, which summarizes our results about the four problems above in a rounded and symmetric form:

**Theorem 1.1.** *The extremality, adjacency, separation, and optimization problems over $P_n$ can all be solved using linear programming without the ellipsoid method.*

A key ingredient in the proof of this theorem is the construction of a novel *extended formulation* for $P_n$, namely, a polynomial time constructible polytope $Q_n$ of size polynomial in $n$ such that $P_n$ is the projection of $Q_n$. Thus, the second main outcome of this paper is the following result which is of interest in its own right.

**Theorem 1.2.** *There exists a polynomial time constructible polytope $Q_n$ of size polynomial in n providing an* extended formulation *for $P_n$, that is, satisfying*

$$P_n = \{x : \exists y \ (x, y) \in Q_n\}.$$

We believe that the advantage of our approach and of the extended formulation in Theorem 1.2 is that, dealing with the set of integer partitions as a polytope and as a projection of another polytope of a polynomial sized description, helps advance the understanding of the geometric structure of the set of partitions, and opens possibilities to apply well-developed polyhedral methods to the study of partitions.

In Sections 2 and 3 we prove Theorems 1.2 and 1.1, respectively. We conclude in Section 4 with some final remarks.

## 2. Lifting the integer partition polytope

In this section we construct a polytope $Q_n$ given by an explicit inequality description of size polynomial in $n$, such that $P_n$ is a projection of $Q_n$. As the first step of this construction we construct a digraph $G_n$ with two distinguished vertices $v_0^0, v_n^n$ such that there is a bijection between integer partitions $x \in T_n$ and $v_0^0 - v_n^n$ dipaths in $G_n$. This construction is inspired by $P_n$ being a knapsack type polytope and the connection between the knapsack problem and dynamic programming [6].

The digraph $G_n = (V, E)$ has the vertex set $V = V^0 \uplus V^1 \uplus \cdots \uplus V^n$ with $V^0 := \{v_0^0\}$, $V^k := \{v_0^k, v_1^k, \ldots, v_n^k\}$ for $k = 1, \ldots, n-1$, and $V^n := \{v_n^n\}$. There are arcs only between consecutive sets $V^{k-1}, V^k$, where an arc $(v_i^{k-1}, v_j^k)$ is included into $E$ if and only if $\frac{j-i}{k} \in \mathbb{Z}_+$. Note that $|V| = \mathcal{O}(n^2)$ and $|E| = \mathcal{O}(n^3)$. Fig. 1 displays $G_6$ for example.

**Lemma 2.1.** *Integer partitions $x \in T_n$ are in bijection with $v_0^0 - v_n^n$ dipaths in $G_n$.*

**Proof.** Given $x \in T_n$, consider any $k$, $1 \le k \le n$; let $i(k) := \sum_{r=1}^{k-1} rx_r$ and let $j(k) := \sum_{r=1}^{k} rx_r$; include in the dipath the arc $(v_{i(k)}^{k-1}, v_{j(k)}^k)$, which exists in $G_n$ since $\frac{j(k)-i(k)}{k} = x_k \in \mathbb{Z}_+$. Since $i(1) = 0$ and $j(n) = n$, this results in a $v_0^0 - v_n^n$ dipath. Conversely, given a $v_0^0 - v_n^n$ dipath, consider any $k$, $1 \le k \le n$; let $(v_{i(k)}^{k-1}, v_{j(k)}^k)$ be the unique arc from $V^{k-1}$ to $V^k$ on that dipath; set $x_k := \frac{j(k)-i(k)}{k} \in \mathbb{Z}_+$. Then $x \in T_n$ since $i(1) = 0, j(k) = i(k+1)$ for $k = 1, \ldots, n-1$, and $j(n) = n$, and hence

$$\sum_{k=1}^{n} kx_k = \sum_{k=1}^{n} j(k) - i(k) = (j(n) - i(n)) + \sum_{k=1}^{n-1} i(k+1) - i(k) = j(n) - i(1) = n. \quad \square$$

Next we define two polytopes $D_n$ and $Q_n$ with certain properties such that $Q_n$ is a suitable lifting of $D_n$ and $P_n$ is a suitable projection of $Q_n$. This is inspired by the polyhedral methods for dynamic programming in [7].

We begin with $D_n$ which is a polytope with $0-1$ vertices standing in bijection with $v_0^0 - v_n^n$ dipaths in $G_n$. For this, we assign to each arc $(v_i^{k-1}, v_j^k)$ in $G_n$ a corresponding $0-1$ variable $y_{i,j}^k$, and we arrange all the arc variables in a vector

$$y = \left(y_{i,j}^k \ : \ k = 1, \ldots, n, \frac{j-i}{k} \in \mathbb{Z}_+\right) \in \{0, 1\}^{|E|}.$$