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Note

The fast robber on interval and chordal graphs

Abbas Mehrabian

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON, Canada

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ABSTRACT

We study a variant of the Cops and Robber game in which the robber has unbounded speed, i.e. can take any path from her vertex in her turn, but she is not allowed to pass through a vertex occupied by a cop. Let $c_\infty(G)$ denote the number of cops needed to capture the robber in graph G in this variant. We show that if G is an interval graph, then $c_\infty(G) \in O(\sqrt{|V(G)|})$; while for every n there exists an n-vertex chordal graph G with $c_\infty(G) \in \Omega(n/\log n)$. This indicates a large contrast between interval graphs and chordal graphs in this variant, whereas in the original Cops and Robber game a single cop suffices for both classes.

1. Introduction

All graphs in this paper are finite and simple. The game of *Cops and Robber* is a perfect information game, played in a graph G. The players are a set of cops and a robber. Initially, the cops are placed at vertices of their choice in G, where more than one cop can be placed at a vertex. Then the robber, being fully aware of the cops' placement, positions herself at one of the vertices of G. Then the cops and the robber move in alternate rounds, with the cops moving first; however, players are permitted to remain stationary in their turn if they wish. The players use the edges of G to move from vertex to vertex. The cops win, and the game ends, if at some point a cop and the robber occupy the same vertex; otherwise, that is, if the robber can elude the cops forever, the robber wins.

This game was defined (for one cop) by Winkler and Nowakowski [16] and Quilliot [17], and has been studied extensively, see Hahn [11] or Bonato and Nowakowski [3]. The best known open question in this area is Meyniel's conjecture, published by Frankl [8], which states that for every connected graph on n vertices, $O(\sqrt{n})$ cops are sufficient to capture the robber. A generalization of this conjecture was proposed in [14]. One intriguing fact about the Cops and Robber game is that although many scholars have studied the game, it is not yet well understood. In particular, although the upper bound $O(\sqrt{n})$ was conjectured already in 1987, no upper bound better than $n^{1-o(1)}$ has been proved since then (see [9,13,18]). In this paper n always denotes the number of vertices of the graph in which the game is played. We consider only connected graphs, since the cop number of a disconnected graph equals the sum of the cop numbers of each connected component.

One might try to change the rules of the game slightly in order to get a more approachable problem, and/or to understand what quality of the original game causes the difficulty. Thus various variations of the game have been studied [2,5,9,12]. The approach chosen by Fomin, Golovach, Kratochvíl, Nisse, and Suchan [7] is to allow the robber move faster than the cops. Inspired by their work, in this paper we let the robber take *any path* from her current position in her turn, but she is not allowed to pass through a vertex occupied by a cop. The parameter of interest is the *cop number* of G, which is defined as the minimum number of cops needed to ensure that the cops can win. We denote the cop number of G by $c_\infty(G)$, where the subscript ∞ indicates that the robber has unbounded speed.

E-mail address: amehrabi@uwaterloo.ca.

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This variant was first studied by Fomin, Golovach, and Kratochvíl [6]. They proved that computing $c_\infty(G)$ is an NP-hard problem, even if G is a split graph. (A *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set.) Next Gavenčiak [10] gave a polynomial time algorithm for interval graphs. This variant was further studied by Frieze, Krivelevich and Loh [9], who showed that for each n, there exists a graph with cop number $\Theta(n)$. Graphs with cop number one were characterized in [15], where lower bounds for cop numbers of expander graphs and Cartesian products of graphs were also proved. Alon and the author [1] determined the asymptotic value of cop number of Erdős–Rényi random graphs, and proved that the cop number of a planar graph is within constant factors of its treewidth. Let us remark that this variant is allied to the so-called graph searching problems (see Bonato and Yang [4]). In particular, a strong connection with the Helicopter Cops and Robber game of Seymour and Thomas [19] is proved in [1].

In this paper we study this game of "Cops and infinitely fast Robber" on interval graphs and chordal graphs. Note that it follows from [11, Theorem 2.4] that one cop can capture the robber in an interval or chordal graph in the original Cops and Robber game. (An easy way to see this is by considering a tree decomposition of the graph in which each bag induces a clique.) However, these two classes are significantly different in the fast robber variant. We show that if G is an interval graph, then $c_\infty(G) \in O(\sqrt{n})$ and provide examples for which this bound is tight. On the other hand, we prove that for every n there exists a chordal graph G with $c_\infty(G) \in \Omega(n/\log n)$. This shows an interesting contrast between interval graphs and chordal graphs in the context of pursuit–evasion games. It remains open to determine whether there are chordal graphs with cop number $\Theta(n)$.

We include some definitions here. For a subset A of vertices, the *neighbourhood* of A, written $N(A)$, is the set of vertices that have a neighbour in A. The *closed neighbourhood* of A, written $\overline{N}(A)$, is the (not necessarily disjoint) union $A \cup N(A)$. If $A = \{v\}$, then we may write $N(v)$ and $\overline{N}(v)$ instead of $N(A)$ and $\overline{N}(A)$, respectively. A *dominating set* is a subset A of vertices with $V(G) = \overline{N}(A)$, and the *domination number* of G is the minimum size of a dominating set of G. The subgraph induced by A is written $G[A]$, and the subgraph induced by $V(G) \setminus A$ is written $G - A$.

2. Interval graphs

Graph G is an *interval graph* if there is a bijection between its vertices and a set of closed intervals on the real line, such that two vertices are adjacent in G if and only if their corresponding intervals intersect. Let G be an interval graph. In this section we prove that $c_\infty(G) \in O(\sqrt{n})$ and that this bound is tight.

Definition (*k-wide*)**.** For a non-empty subgraph H of G, say H is *k-wide* if

(i) H is k-connected, and
(ii) for any $S \subseteq V(G)$ with $|S| < k$ we have $V(H) \not\subseteq \overline{N}(S)$.

Lemma 2.1. *If G has a k-wide subgraph H then $c_\infty(G) \ge k$.*

Proof. Say a cop *controls* a vertex u if the cop is at u or at an adjacent vertex. Suppose that there are fewer than k cops in the game, and that they initially start at a subset S of vertices. By condition (ii), there exists a vertex v in $V(H) \setminus \overline{N}(S)$, i.e. v is controlled by no cop. The robber starts at v, and will always remain in H. After each move of the cops, the set of vertices occupied by them has size smaller than k. Hence by condition (ii), there exists a vertex x of H that is not controlled by any cop. By condition (i), H is k-connected, so as the robber is currently in H, and there are fewer than k cops, there is a cop-free path to x. The robber moves there and will not be captured in the next round. Since she can elude forever by using this strategy, at least k cops are needed to capture her. □

Consider a set of closed intervals whose intersection graph is G, and denote by I_v the interval corresponding to vertex v. We may assume without loss of generality that none of the intervals have zero length. Let $x_1 < x_2 < \cdots < x_{l+1}$ be the set of distinct endpoints of the intervals, and let $y_1, y_2, \ldots, y_l$ be points satisfying $x_i < y_i < x_{i+1}$ for each $1 \le i \le l$. Also, define $V_i = \{v \in V(G) : y_i \in I_v\}$ for $1 \le i \le l$. It is clear that each $G[V_i]$ is a clique. Furthermore, $l \le 2n$ and the sets $V_1, \ldots, V_l$ cover the vertices of G.

Say $A \subseteq V(G)$ is a *cut-set* of G if $G - A$ has more connected components than G.

Lemma 2.2. *Every minimal cut-set X of G is one of the V_i's. Moreover, if $X = V_i$ is a cut-set, then for each $u_1 \in V_{i_1} \setminus X$ and $u_2 \in V_{i_2} \setminus X$ satisfying $i_1 < i < i_2$, u_1 and u_2 lie in different components of $G - X$.*

Proof. For an index $1 \le i \le l$, say point y_i is a *cut-point* if there exists a vertex $v \in V(G)$ with both endpoints of I_v lying strictly on the left of y_i, and also a vertex $v' \in V(G)$ with both endpoints of $I_{v'}$ lying strictly on the right of y_i. If y_i is a cut-point then clearly V_i is a cut-set of G.

Now, let X be a minimal cut-set of G. Let u_1, u_2 be vertices in different components of $G - X$, with $I_{u_1} = [x_a, x_b]$, $I_{u_2} = [x_c, x_d]$, and assume by symmetry that $a < b < c < d$. For each i with $b \le i \le c - 1$, y_i is a cut-point. If for all of the i's in this range, there existed a vertex $v_i \in V_i \setminus X$, then $u_1 v_b v_{b+1} \ldots v_{c-1} u_2$ would be a (u_1, u_2)-path in $G - X$. As such a path does not exist, there is an i in this range such that $V_i \subseteq X$. But then V_i is a cut-set of G, hence $X = V_i$.

For the second statement, let $X = V_i$ be a cut-set, $u_1 \in V_{i_1} \setminus X$ and $u_2 \in V_{i_2} \setminus X$ such that $i_1 < i < i_2$. Let $I_{u_1} = [x_a, x_b]$, $I_{u_2} = [x_c, x_d]$, and so $x_a < x_b < y_i < x_c < x_d$. Every (u_1, u_2)-path contains a vertex whose corresponding interval contains y_i, but all such vertices are in X. Hence there is no (u_1, u_2)-path in $G - X$. □