Discrete Applied Mathematics 158 (2010) 198–203

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Series–parallel chromatic hypergraphs

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ARTICLE INFO

Article history:
Received 23 September 2008
Received in revised form 29 June 2009
Accepted 2 September 2009
Available online 1 October 2009

Keywords:
Two-terminal series–parallel chromatic hypergraph
Multibridge hypergraph
Linear h-uniform cycle
Chromatic polynomial
Chromatic uniqueness

ABSTRACT

In this paper two-terminal series–parallel chromatic hypergraphs are introduced and for this class of hypergraphs it is shown that the chromatic polynomial can be computed with polynomial complexity. It is also proved that h-uniform multibridge hypergraphs $\theta(h; a_1, a_2, \ldots, a_k)$ are chromatically unique for $h \geq 3$ if and only if $h = 3$ and $a_1 = a_2 = \cdots = a_k = 1$, i.e., when they are sunflower hypergraphs having a core of cardinality 2 and all petals being singletons.

1. Notation and preliminary results

A *simple hypergraph* $H = (V, \mathcal{E})$ having *order* $n = |V|$ and *size* $m = |\mathcal{E}|$ consists of a *vertex-set* $V(H) = V$ and an *edge-set* $E(H) = \mathcal{E}$, where $E \subseteq V$ and $|E| \geq 2$ for each edge $E \in \mathcal{E}$. If $\mathcal{E}$ is a multiset, H will be called a *multihypergraph*. For $h \geq 2$, H is said to be *h-uniform*, or an h-hypergraph, if $|E| = h$ for each $E \in \mathcal{E}$ and H is *linear* if no two edges intersect at more than one vertex. A hypergraph for which no edge is a subset of any other is called *Sperner*.

The number of edges containing a vertex x is its *degree* $d_H(x)$. H is said to be *connected* if for any two vertices $u, v \in V(H), u \neq v$, there are vertices $x_0 = u, x_1, \ldots, x_k = v$ and edges $E_1, \ldots, E_k \in E(H)$ such that $x_{i-1}, x_i \in E_i$ for each i, $1 \leq i \leq k$.

A *cycle* C of *length* k in H [1] is a subhypergraph comprising k distinct vertices $x_1, \ldots, x_k$ and k distinct edges $E_1, \ldots, E_k$ of H such that $x_i, x_{i+1} \in E_i$ for each i, $1 \leq i \leq k-1$ and $x_1, x_k \in E_k$. C is said to be *elementary* if $d_C(x_i) = 2$ for each i and $d_C(y) = 1$ for each other vertex y in $\bigcup_{i=1}^{k} E_i$.

We shall denote the elementary h-uniform cycle with m edges by C_m^h; clearly it is linear and has order $m(h-1)$.

An h-uniform *hypertree* is a connected linear h-hypergraph without cycles.

For any $a_1, a_2, \ldots, a_k \in \mathbb{N}$ and $h \geq 2$ we denote by $\theta(h; a_1, a_2, \ldots, a_k)$ the h-uniform linear hypergraph consisting of k h-uniform linear paths of length $a_1, a_2, \ldots, a_k$, joined in parallel and having only two vertices s and t in common. $\theta(4; 3, 3, 2)$ is illustrated in Fig. 1. $\theta(h; a_1, a_2, \ldots, a_k)$ will be called a *multibridge* hypergraph (or, more precisely, a *k-bridge hypergraph*). Note that $\theta(2; a_1, a_2, \ldots, a_k)$ is called a multibridge graph [7].

If $\lambda \in \mathbb{N}$, a λ-*coloring* of a hypergraph H is a function $f : V(H) \to \{1, \ldots, \lambda\}$ such that for each edge E of H there exist x, y in E such that $f(x) \neq f(y)$. The number of λ-colorings of H is given by a polynomial $P(H, \lambda)$ of degree $|V(H)|$ in λ, called the *chromatic polynomial* of H, whose coefficients are described in the following lemma.

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0166-218X/$ – see front matter
doi:10.1016/j.dam.2009.09.003

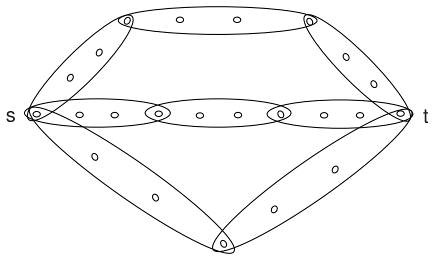

Fig. 1. $\theta(4; 3, 3, 2)$.

Lemma 1.1 (*[10]*). *Let H be a hypergraph of order n. Then $P(H,\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda$, where $a_i = \sum_{j\geq 0}(-1)^j N(i,j)$ $(1 \leq i \leq n-1)$ and $N(i,j)$ denotes the number of subhypergraphs of H with n vertices, i components and j edges.*

Chromatic polynomials of elementary h-uniform cycles and of h-uniform hypertrees are given by the following lemma.

Lemma 1.2 ([6]). *If C_m^h is an elementary h-uniform cycle with m edges and T_k^h is any h-uniform hypertree with k edges then $P(C_m^h,\lambda) = (\lambda^{h-1}-1)^m + (-1)^m(\lambda-1)$ and $P(T_k^h,\lambda) = \lambda(\lambda^{h-1}-1)^k$.*

Two hypergraphs H and G are said to be *chromatically equivalent*, written as $H \sim G$, if $P(H,\lambda) = P(G,\lambda)$. In the class of Sperner hypergraphs a simple hypergraph H is said to be *chromatically unique* (or *χ-unique*) if H is isomorphic to H' for every simple hypergraph H' such that $H' \sim H$. This means that the structure of H is uniquely determined up to isomorphism by its chromatic polynomial. The notion of χ-unique graphs was first introduced and studied by Chao and Whitehead [5].

For $h \geq 3$ let $SH(n,p,h)$ denote the h-hypergraph H (unique up to isomorphism) defined as follows [11]: $|V(H)| = n = h + (k-1)p$ $(1 \leq p \leq h-1)$, $|\mathcal{E}(H)| = k$ and there exist a "core" $X \subset V(H)$, $|X| = h-p$ and an equipartition of $V(H) \setminus X$ into k "petals": $V(H) \setminus X = Y_1 \cup \cdots \cup Y_k$, where $|Y_1| = \cdots = |Y_k| = p$ such that $\mathcal{E}(H) = (X \cup Y_i)_{1\leq i\leq k}$. $SH(n,p,k)$ is called the *sunflower hypergraph*. Note that this terminology goes back to Erdős and Rado [9].

The following two theorems explain the chromaticity of sunflower hypergraphs.

Theorem 1.1 (*[3]*). *$SH(n,1,h)$ is chromatically unique.*

Theorem 1.2 (*[11]*). *$SH(n,p,h)$ is not chromatically unique for every $p, k \geq 2$.*

In [2] it was shown that C_m^h is not chromatically unique for $m = 3$; in the next section we shall prove that this property is true for every cycle of length $m \geq 3$.

Now we shall define the notion of two-terminal series–parallel chromatic hypergraph, which extends the notion of the two-terminal series–parallel graph (see e.g. [4]).

A *two-terminal chromatic hypergraph* is a quintuple $(H, s, t, \varphi_1, \varphi_2)$, where $H = (V, E)$ is a multihypergraph, $s, t \in V$, $s \neq t$ (s is called the *source* and t is called the *sink* of H) and φ_1, φ_2 are two functions of the variable λ.

The *series composition* of the two-terminal chromatic hypergraphs $((V_1, E_1), s_1, t_1, \varphi_1^1, \varphi_2^1)$ and $((V_2, E_2), s_2, t_2, \varphi_1^2, \varphi_2^2)$ with $t_1 = s_2$ and $V_1 \cap V_2 = \{t_1\}$ is the two-terminal chromatic hypergraph $((V_1 \cup V_2, E_1 \cup E_2), s_1, t_2, \varphi_1, \varphi_2)$, where

$$\varphi_1 = \frac{1}{\lambda}\left(\varphi_1^1\varphi_1^2 + \frac{1}{\lambda-1}\varphi_2^1\varphi_2^2\right); \qquad \varphi_2 = \frac{1}{\lambda}\left(\varphi_1^1\varphi_2^2 + \varphi_2^1\varphi_1^2 + \frac{\lambda-2}{\lambda-1}\varphi_2^1\varphi_2^2\right). \tag{1}$$

The *parallel composition* of two-terminal chromatic hypergraphs $((V_1, E_1), s_1, t_1, \varphi_1^1, \varphi_2^1)$ and $((V_2, E_2), s_2, t_2, \varphi_1^2, \varphi_2^2)$ with $s_1 = s_2$ and $t_1 = t_2$ and $V_1 \cap V_2 = \{s_1, t_1\}$ is the two-terminal chromatic hypergraph $((V_1 \cup V_2, E_1 \cup E_2), s_1, t_1, \varphi_1, \varphi_2)$, where

$$\varphi_1 = \frac{1}{\lambda}\varphi_1^1\varphi_1^2; \qquad \varphi_2 = \frac{1}{\lambda(\lambda-1)}\varphi_2^1\varphi_2^2. \tag{2}$$

We shall propose a recursive definition of a two-terminal series–parallel (for short, SP) chromatic hypergraph. The two-terminal hypergraph $(H, s, t, \varphi_1, \varphi_2)$ is a *two-terminal SP chromatic hypergraph* if it consists of only one edge E with two distinct vertices $s, t \in E$ as source and sink and

$$\varphi_1 = \lambda(\lambda^{h-2}-1); \qquad \varphi_2 = (\lambda-1)\lambda^{h-1}, \tag{3}$$

where $h = |E|$ (a so-called *basic* two-terminal SP hypergraph) or it results from the application of the series or the parallel composition of two-terminal SP chromatic hypergraphs. Note that if some basic two-terminal SP chromatic hypergraphs have cardinality 2, by parallel composition of these edges there may appear parallel edges of cardinality 2, and the resulting two-terminal SP chromatic hypergraphs are multihypergraphs. The chromatic meaning of the functions φ_1 and φ_2 will be explained in the next section.