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Rainbow connection in oriented graphs[☆]

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ABSTRACT

An edge-coloured graph G is said to be *rainbow-connected* if any two vertices are connected by a path whose edges have different colours. The rainbow connection number of a graph is the minimum number of colours needed to make the graph rainbow-connected. This graph parameter was introduced by G. Chartrand, G.L. Johns, K.A. McKeon and P. Zhang in 2008. Since, the topic drew much attention, and various similar parameters were introduced, all dealing with undirected graphs.

Here, we initiate the study of rainbow connection in oriented graphs. An early statement is that the rainbow connection number of an oriented graph is lower bounded by its diameter and upper bounded by its order. We first characterize oriented graphs having rainbow connection number equal to their order. We then consider tournaments and prove that (i) the rainbow connection number of a tournament can take any value from 2 to its order minus one, and (ii) the rainbow connection number of every tournament with diameter d is at most $d + 2$.

1. Introduction

We consider finite and simple graphs only, and refer to [1] for terminology and notations not defined here.

In an edge-coloured graph G, a path is said to be *rainbow* if it does not use two edges with the same colour. Then the graph G is said to be *rainbow-connected* if any two vertices are connected by a rainbow path. This concept of rainbow connection in graphs was recently introduced by Chartrand et al. in [4]. An application of rainbow connection for the secure transfer of classified information between agencies in communication networks was presented in [5]. Along with it, rainbow paths are generally used in the concept of onion routing, using layered encryption [12]. For onion routing, one enciphers a message once by hop on the path, always with different keys (corresponding to the colours of the edges). This layered encryption is used e.g. by the anonymous networks TOR and I2P.

In the following, we are interested in the corresponding optimization parameter. The *rainbow connection number* of a connected graph G, denoted by $rc(G)$, is the smallest number of colours that are needed in order to make G rainbow connected. The computational complexity of rainbow connectivity was studied in [3], where it is proved that the computation of $rc(G)$ is NP-hard. In fact it is already NP-complete to decide if $rc(G) = k$ for any fixed $k \geq 2$ or to decide whether a given edge-coloured (with any number of colours) graph is rainbow connected.

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Additionally, Chartrand et al. computed the precise rainbow connection number of several graph classes including complete multipartite graphs [4]. The rainbow connection number was studied for further graph classes in [2,6,7,9,10,14] and for graphs with fixed minimum degree in [2,8,13,15]. Also, different other parameters similar to rainbow connection were introduced such as strong rainbow connection, rainbow k-connectivity, k-rainbow index and rainbow vertex connection. See [9] for a survey about these different parameters.

In this paper, we extend the problem of rainbow connection to oriented graphs. Whereas it was easily observed that a graph of order n has rainbow connection number at most $n-1$ (giving different colours to all the edges in a spanning tree), the rainbow connection number of an oriented graph can be equal to its order. In this paper, we characterize oriented graphs with rainbow connection number equal to their order.

We proceed as follows. We start with useful definitions in Section 2. Then, in Section 3, we prove that the only minimally strong oriented graphs that have rainbow connection number exactly their order are cycles. In Section 4, we propose a characterization of all oriented graphs with rainbow connection number equal to their order. Finally, we prove in Section 5 that the rainbow connection number of a tournament can take almost any value in terms of its order, but is upper bounded by the tournament diameter plus 2.

2. Definitions, notation and basic results

2.1. Definitions and notation

For a given digraph G, we denote by $V(G)$ and $A(G)$ respectively its sets of vertices and of arcs. By an *oriented graph* we mean an antisymmetric digraph, that is where $yx \notin A(G)$ whenever $xy \in A(G)$. Given an arc xy in G, we say y is an *out-neighbour* of x while x is an *in-neighbour* of y. Moreover, we call x the *tail* of xy and y the *head* of xy.

By $N^+_G(x)$ (resp. $N^-_G(x)$) we denote the set of out-neighbours (resp. in-neighbours) of x in G. The *out-degree* (resp. the *in-degree*) of x is the order of its out-neighbourhood $d^+_G(x) = |N^+_G(x)|$ (resp. in-neighbourhood $d^-_G(x) = |N^-_G(x)|$), and the *degree* of x is simply $d_G(x) = d^+_G(x) + d^-_G(x)$.

For X a subset of $V(G)$, we denote by $G[X]$ the subgraph of G *induced by* X, given by $V(G[X]) = X$ and $A(G[X]) = A(G) \cap (X \times X)$. A *spanning subgraph* H of G is a subgraph of G with $V(H) = V(G)$.

A *path* of length $k \geq 1$ in an oriented graph G is a sequence $x_0 \dots x_k$ of vertices such that $x_i x_{i+1} \in A(G)$ for every i, $0 \leq i \leq k-1$. Such a path P, going from x_0 to x_k, is referred to as an $(x_0 - x_k)$*-path*. Any vertex in $V(P) \setminus \{x_0, x_k\}$ is an *internal vertex* of P. If X and Y are two subsets of $V(G)$, an $(X - Y)$*-path* is an $(x - y)$-path linking a vertex $x \in X$ to a vertex $y \in Y$. A path P is *elementary* if no vertex appears twice in P. An elementary path *induced by* a path Q is any elementary path P obtained from Q by repeatedly deleting cycles, that is replacing a sequence of the form $u_1 \dots u_k x v_1 \dots v_\ell x w_1 \dots w_m$ by $u_1 \dots u_k x w_1 \dots w_m$ as many times as necessary. Given two paths $P_1 = x_1 \dots x_i$ and $P_2 = x_i \dots x_{i+j}$, we denote by $P_1 \cup P_2$ the path $x_1 \dots x_i \dots x_{i+j}$.

An *ear* in an oriented graph G is an $(x-y)$-path Q such that $d_G(x) > 2$, $d_G(y) > 2$ and $d_G(z) = 2$ for every internal vertex z of Q.

The *distance* from a vertex x to a vertex y in an oriented graph G, denoted by $\text{dist}_G(x, y)$, is the length of a shortest $(x-y)$-path in G (if there is no such path, we say $\text{dist}_G(x, y) = \infty$). The *diameter* of G, denoted by $\text{diam}(G)$, is the maximum distance between any pair of vertices in G. Two vertices at distance $\text{diam}(G)$ are *antipodal vertices*. The *eccentricity* of a vertex x in G, denoted by $\text{ecc}_G(x)$, is the maximum distance from x to any other vertex y of G.

Let G be an oriented graph. For an arc xy in $A(G)$, we denote by $G-xy$ the oriented graph defined by $G-xy = (V(G), A(G) \setminus \{xy\})$. For a vertex u in $V(G)$, we denote by $G-u$ the oriented graph $G-u = (V(G) \setminus \{u\}, (A(G) \setminus (\{u\} \times V(G)) \setminus (V(G) \times \{u\})))$. For G' an oriented graph, we denote by $G \cup G'$ the oriented graph given by $G \cup G' = (V(G) \cup V(G'), A(G) \cup A(G'))$.

An oriented graph G is *strongly connected* (*strong* for short) if there exists an $(x - y)$-path in G for every two vertices x and y. The graph G is *minimally strongly connected* (MSC for short) if G is strong and, for every arc xy in G, the graph $G - xy$ is not strong.

A *cycle* of length $k \geq 3$ in an oriented graph G is a sequence $x_0 \dots x_{k-1} x_0$ of vertices such that $x_i x_{i+1} \in A(G)$ for every i, $0 \leq i \leq k-2$, and $x_{k-1} x_0 \in A(G)$. For every i, $0 \leq i \leq k-1$, x_{i+1} (resp. x_{i-1}) is the *successor* (resp. *predecessor*) of x_i in C (subscripts are taken modulo k).

2.2. Rainbow connection of oriented graphs

Let G be an oriented graph. A k*-arc-colouring* of G, $k \geq 1$, is a mapping $\varphi : A(G) \to \{1, \dots, k\}$. Note that adjacent arcs may receive the same colour. An *arc-coloured* oriented graph is then a pair (G, φ) where G is an oriented graph and φ an arc-colouring of G. A path P in (G, φ) is *rainbow* if no two arcs of P are coloured with the same colour. An arc-coloured oriented graph (G, φ) is *rainbow connected* (or, equivalently, φ is a *rainbow arc-colouring* of G) if any two vertices in G are connected by a rainbow path. Note that in order to admit a rainbow arc-colouring, an oriented graph must be strong.

The *rainbow connection number* of an oriented graph G, denoted by $\vec{rc}(G)$, is defined as the smallest number k such that G admits a rainbow k-arc-colouring.

Note that in a rainbow connected graph, there must be a path with at least $\text{diam}(G)$ colours between antipodal vertices. We thus have the following proposition.