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On the bend-number of planar and outerplanar graphs

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ABSTRACT

The *bend-number* $b(G)$ of a graph G is the minimum k such that G may be represented as the edge intersection graph of a set of grid paths with at most k bends. We confirm a conjecture of Biedl and Stern showing that the maximum bend-number of outerplanar graphs is 2. Moreover we improve the formerly known lower and upper bounds for the maximum bend-number of planar graphs from 2 and 5 to 3 and 4, respectively.

1. Introduction

In [17] Golumbic, Lipshteyn and Stern defined an *EPG*[1] representation of a simple graph G as an assignment of paths in the rectangular plane grid to the vertices of G, such that two vertices are adjacent if and only if the corresponding paths intersect in at least one grid edge, see Fig. 1 for an example.

EPG representations arise from VLSI grid layout problems [7] and as generalizations of *edge-intersection graphs of paths on degree 4 trees* [16]. In the same paper Golumbic et al. show that every graph has an EPG representation and propose to restrict the number of bends per path in the representation. There has been some work related to this, see [3,17,2,4,26]. A graph is a *k-bend graph* if it has an EPG representation, where each path has at most k bends. The *bend-number* $b(G)$ of G is the minimum k, such that G is a k-bend graph.

Note that the class of 0-bend graphs coincides with the well-known class of interval graphs, i.e., intersection graphs of intervals on a real line. Interval graphs have many nice properties, in particular they are recognizable in linear time [6]. It is thus natural to view k-bend graphs as an extension of the concept of interval graphs and the bend-number as a measure of how far a graph is from being an interval-graph. Notably, in [26] it is shown that already recognition of graphs of bend-number at most 1 is NP-complete. The recognition complexity was then further investigated for subclasses of 1-bend graphs in [8]. Further algorithmic questions concerning 1-bend graphs have been addressed in [14].

There are other ways to measure how far a given graph is from being an interval graph. Intersection graphs of systems of intervals together with a parameter counting how many intervals are needed to represent a vertex of a given graph G have received some attention:

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[1] EPG stands for *edge intersection graph of paths in the grid*.

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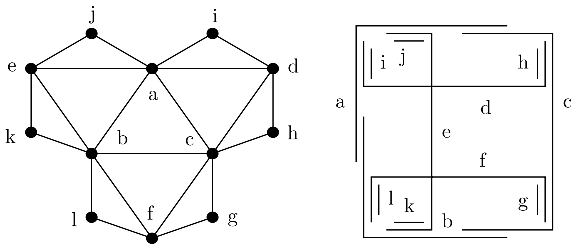

Fig. 1. A 2-bend graph and an EPG representation. If a grid edge is shared by several paths we draw them close to each other.

Table 1
Some graph classes and their maximum interval-number, track-number and bend-number. Here dg(G), tw(G), and $\Delta(G)$ denote degeneracy, treewidth and maximum degree of G, respectively.

	$i(G)$		$t(G)$		$b(G)$	
Forest	2	[24]	2	[11]	1	[17]
Outerplanar	2	[31]	2	[28]	2	Proposition 2
Planar	3	[31]	4	[18]	$3 \leq . \leq 4$	Theorem 9
+ bipartite	3	[31]	4	[18,19]	2	[4]
Line graph	2		?	Conjecture [26]	2	[4]
dg(G) $\leq k$	$k+1$	[27]	$2k$	[1,23,27]	$2k-1$	[26]
tw(G) $\leq k$	$k+1$	[12,27]	$k+1$	[12,27]	$2k-2$ $(k \geq 3)$	[26]
$\Delta(G) \leq k$	$\lceil \frac{k+1}{2} \rceil$	[20]	$\leq \frac{3k+6}{5}$	[21]	$\lceil \frac{k}{2} \rceil \leq . \leq k$	[26]

- in a *k-interval representation* of a graph G every vertex is associated with a set of at most k intervals on the real line, such that vertices are adjacent iff any of their intervals intersect. The *interval-number* $i(G)$ is then defined as the minimum k, such that G has a k-interval representation, see [24];
- in a *k-track representation* of a graph G there are k parallel lines, called *tracks*. Every vertex is associated with one interval from each track. Again vertex adjacency is equivalent to interval intersection and the *track-number* $t(G)$ is the minimum k, such that G has a k-track representation, see [22].

Now, $b(G)$ can be set in relation to $i(G)$ and $t(G)$: $b(G)$ is only a constant factor away from $i(G)$ and $t(G)$. Consecutive intervals representing a vertex in a k-interval representation may be connected by introducing three segments such that they form a grid-path. It is easy to see, that a k-track representation can be transformed into a k-interval representation, by putting the tracks on a single line, i.e., $i(G) \leq t(G)$. Thus one gets $b(G) \leq 4(i(G)-1) \leq 4(t(G)-1)$. On the other hand the grid-lines of a k-bend representation can be stringed together on a single line, i.e., $i(G) \leq b(G)+1$.

Extremal questions for these parameters like *'What is the maximum interval-/track-/bend-number among all graphs of a particular graph class?'* have been of strong interest in the literature. In Table 1 we give an overview of some results in this spirit.

In [4] Biedl and Stern show that outerplanar graphs are 3-bend graphs and provide an outerplanar graph which has bend-number 2, see Fig. 1. They conjecture that all outerplanar graphs are 2-bend graphs. We confirm this conjecture in Theorem 1, showing the stronger result that graphs of treewidth at most 2 are 2-bend graphs.

The major part of this paper is devoted to planar graphs. Biedl and Stern [4] show that planar graphs are 5-bend graphs but the only lower bound that they have is 2, given by the graph of Fig. 1. In Proposition 4 we provide a planar graph of treewidth 3 which has bend-number 3, thus improving the lower bound of the class of planar graphs by one. Indeed in Theorem 3 we show that every planar graph with treewidth 3 is a 3-bend graph. The main result of this article is Theorem 8. We improve the upper bound for the bend-number of general planar graphs from 5 to 4.

All our upper bounds are achieved by explicit constructions and can be easily transferred into polynomial-time algorithms computing the desired k-bend representation of the planar graph in question. Note that every k-bend path is uniquely determined by a sequence of $k+2$ grid points (2 ends and k bends), and that whether or not two such paths share a grid edge only depends on the left-to-right and top-to-bottom partial order of these grid points. Since every k-bend path uses at most $\lceil (k+3)/2 \rceil$ horizontal and $\lceil (k+3)/2 \rceil$ vertical grid lines, every k-bend representation of an n-vertex graph fits on the $(\lceil (k+3)/2 \rceil n) \times (\lceil (k+3)/2 \rceil n)$ grid. Moreover, instead of defining grid paths by explicit coordinates, in our constructive proofs it suffices to describe each path by its sequence of grid points and give the left-to-right order and top-to-bottom order of all vertical and horizontal grid lines containing an end or bend of some grid path, respectively. In particular, we may at any time introduce between two consecutive horizontal or vertical grid lines a new grid line and route a new grid path on this new line. In the end, we can easily derive from the left-to-right and top-to-bottom order of the grid lines we used a k-bend representation with actual coordinates in $O(kn)$.

An extended abstract of this paper has appeared in the proceedings of LATIN 2012 [25].