Discrete Applied Mathematics 179 (2014) 152–162

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On the event distance of Poisson processes with applications to sensors

Evangelos Kranakis *

School of Computer Science, Carleton University, Ottawa, ONT, K1S5B6, Canada

ARTICLE INFO

Article history:
Received 2 December 2013
Received in revised form 28 June 2014
Accepted 20 July 2014
Available online 6 August 2014

Keywords:
Arrival time
Gamma distribution
Event distance
Matchings
Poisson
Transportation cost

ABSTRACT

We derive a closed formula for the expected distance between Poisson events of two i.i.d. Poisson processes with arrival rate λ and respective arrival times $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ Namely, for any integers $r \geq 0$, $k \geq 1$, the following identity holds:

$$E[|X_{k+r} - Y_k|] = \frac{k2^{-2k+1}}{\lambda}\binom{2k}{k}\left(1 + \sum_{s=0}^{r-1}\frac{r-s}{(2k+s)2^s}\cdot\frac{(2k+1)^{(s)}}{(k+1)^{(s)}}\right),$$

where $x^{(q)}$ denotes the Pochhammer polynomial. As a consequence we derive that the expected cost of a minimum weight matching with edges $\{X_i, Y_i\}$ between two i.i.d. Poisson processes with arrival times $X_1, X_2, \ldots X_n$ and $Y_1, Y_2, \ldots Y_n$ is in $\Theta(\sqrt{n})$.

1. Introduction

There is interest in the research community for understanding the cost of sensor movement (in a sensor network) measured either as the sum or maximum of movements of sensors from their initial positions towards target destinations, respectively. In this paper we study the sum assuming that the sensors are placed at random on a line according to a Poisson process. Further, we derive a closed formula for the expected distance of events of two i.i.d. Poisson processes.

[1,7] are two papers and [9] a book related to these questions. The optimal transportation cost for random matchings of bicolored point sets was considered in [1] where it is proved that for $2n$ points (n colored 0 and n colored 1) placed at random in the unit square the answer is $\Theta(\sqrt{n \ln n})$. Talagrand in [9, Chapter 3] studies matching theorems for N random variables $X_1, \ldots, X_N$ independently uniformly distributed in the d-dimensional unit cube $[0, 1]^d$, where $d \geq 2$. In [7] the authors consider the expected maximum total (i.e., sum) of movements of n identical sensors placed uniformly at random in a unit interval so as to attain complete coverage of the unit interval [0, 1] and consider tradeoffs between the sensor range and total movement.

In this paper, we study the event distance between two i.i.d. Poisson processes with respective arrival times $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ on a line and derive a closed form formula for the event distances $E[|X_{k+r} - Y_k|]$, for any $r \geq 0$, $k \geq 1$. A typical motivation for studying these quantities could arise in sensor networks with one Poisson process representing resources and the other events and is required to match one-by-one resources with events (see [4]).

* Tel.: +1 613 521 0154; fax: +1 613 520 4334.
E-mail address: kranakis@scs.carleton.ca.

http://dx.doi.org/10.1016/j.dam.2014.07.019

1.1. Preliminaries

In this subsection we introduce some basic concepts and notation that will be used throughout the paper. Let $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt$, $\Gamma(z,a) = \int_a^{+\infty} t^{z-1}e^{-t}dt$, $\gamma(z,a) = \int_0^a t^{z-1}e^{-t}dt$ denote the *Gamma*, *upper* incomplete gamma, and the *lower* incomplete gamma functions, respectively.

If X is a Poisson random variable with mean λ then $\Pr[X = x] = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$. Let X_i be the arrival time of the ith event in a Poisson process with arrival rate λ. If $T_1, T_2, \ldots$ are the interarrival times of the Poisson process then $X_i = T_1 + T_2 + \cdots + T_i$. The random variable X_i obeys the gamma distribution with parameters i, λ. Its probability density function is given by $f_{X_i}(s) = \lambda e^{-\lambda s}\frac{(\lambda s)^{i-1}}{(i-1)!}$ and

$$\Pr[X_i = s] = \lambda e^{-\lambda s}\frac{(\lambda s)^{i-1}}{(i-1)!} \quad \text{and} \quad \Pr[X_i \geq s] = \int_s^{+\infty} \lambda e^{-\lambda t}\frac{(\lambda t)^{i-1}}{(i-1)!}dt.$$

Moreover, $E[X_i] = \frac{i}{\lambda}$ and $\mathrm{Var}(X_i) = \frac{i}{\lambda^2}$. The probability density function of $X_{i+n} - X_i$, i.e. the distance between two Poisson events which are separated by $n-1$ other Poisson events, is given by the formula $\frac{\lambda^n}{(n-1)!}e^{-\lambda x}x^{n-1}$, where $0 \leq x < +\infty$, and the cumulative distribution is $\Pr[X_{i+n} - X_i \leq x] = \frac{\gamma(n,\lambda x)}{(n-1)!}$. (See [8, p. 204] and [6, p. 42] for additional details on the Poisson process.)

Finally, we prove some useful identities involving indefinite and definite integrals that will be used in the proof of Lemma 1. Using integration by parts we derive

$$\int \lambda e^{-\lambda t}\frac{(\lambda t)^{i-1}}{(i-1)!}dt = -e^{-\lambda t}\sum_{j=0}^{i-1}\frac{(\lambda t)^j}{j!}.$$

From this we can derive easily

$$\int_0^{+\infty} \lambda e^{-\lambda t}\frac{(\lambda t)^{i-1}}{(i-1)!}dt = \left[-e^{-\lambda t}\sum_{j=0}^{i-1}\frac{(\lambda t)^j}{j!}\right]_0^{+\infty} = 1$$

$$\int_{y_k}^{+\infty} \lambda e^{-\lambda t}\frac{(\lambda t)^{i-1}}{(i-1)!}dt = \left[-e^{-\lambda t}\sum_{j=0}^{i-1}\frac{(\lambda t)^j}{j!}\right]_{y_k}^{+\infty} = e^{-\lambda y_k}\sum_{j=0}^{i-1}\frac{(\lambda y_k)^j}{j!}.$$

For the next indefinite integral we use substitution to derive the following identity

$$\int t\lambda e^{-\lambda t}\frac{(\lambda t)^{i-1}}{(i-1)!}dt = \frac{i}{\lambda}\int \lambda e^{-\lambda t}\frac{(\lambda t)^i}{i!}dt = -\frac{i}{\lambda}e^{-\lambda t}\sum_{j=0}^{i}\frac{(\lambda t)^j}{j!}.$$

From this we can derive easily

$$\int_0^{+\infty} t\lambda e^{-\lambda t}\frac{(\lambda t)^{i-1}}{(i-1)!}dt = \frac{i}{\lambda}\left[-e^{-\lambda t}\sum_{j=0}^{i}\frac{(\lambda t)^j}{j!}\right]_0^{+\infty} = \frac{i}{\lambda}$$

$$\int_{y_k}^{+\infty} t\lambda e^{-\lambda t}\frac{(\lambda t)^{i-1}}{(i-1)!}dt = \frac{i}{\lambda}\left[-e^{-\lambda t}\sum_{j=0}^{i}\frac{(\lambda t)^j}{j!}\right]_{y_k}^{+\infty} = \frac{i}{\lambda}e^{-\lambda y_k}\sum_{j=0}^{i}\frac{(\lambda y_k)^j}{j!}.$$

1.2. Contributions

A closed form formula (see Theorem 1) is derived for the expected distance between Poisson events of two i.i.d. Poisson processes, namely the following formula is proved

$$E[|X_{k+r} - Y_k|] = \frac{k2^{-2k+1}}{\lambda}\binom{2k}{k}\left(1 + \sum_{s=0}^{r-1}\frac{r-s}{(2k+s)2^s}\cdot\frac{(2k+1)^{(s)}}{(k+1)^{(s)}}\right),$$

for any integers $r \geq 0$, $k \geq 1$, where $x^{(q)}$ denotes the Pochhammer polynomial.

Further, we provide an application to sensor networks concerning the optimal expected movement and transportation cost of sensors on the line $[0, +\infty)$. We consider matchings of bi-colored random point-sets in the line $[0, +\infty)$. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be the first n arrival times of two i.i.d. Poisson processes, respectively, with arrival rate $\lambda = n$, assumed to be the positions of $2n$ sensors placed at random in the interval $[0, +\infty)$ and colored 0 and 1, respectively. We prove that the expected minimum length of a bicolored complete matching (i.e., the vertices of each matching edge have different colors) is in $\Theta(\sqrt{n})$.