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Upper bounds on the balanced $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph

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ABSTRACT

Let $G = (V, E)$ be a simple graph of order n with vertex set $V = \{v_1, \ldots, v_n\}$ and suppose that at most r_i units of some commodity may be placed at any vertex v_i while at least s_i units must be placed in the closed neighbourhood of v_i for $i = 1, \ldots, n$. The smallest number of units that may be placed on the vertices of the graph satisfying the above requirements is called the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the graph. The case where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ is called the balanced case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination. We establish three upper bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph for the balanced case in this paper.

1. Introduction

Let $G = (V, E)$ be a simple graph of order n with vertex set $V = \{v_1, \ldots, v_n\}$ and define the *open neighbourhood* $N(v_i)$ of a vertex $v_i \in V$ as the set $\{v_j : \ v_iv_j \in E\}$ and its *closed neighbourhood* $N[v_i]$ as the set $N(v_i) \cup \{v_i\}$. Furthermore, let d_v denote the degree of the vertex v. A set $S \subseteq V$ is a *dominating set* of G if every vertex in V is either in S or adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

Cockayne [4] introduced a general framework for domination in graphs. Let $\boldsymbol{r}$ and $\boldsymbol{s}$ be non-negative integer n-vectors. As in [4], we define an $\boldsymbol{r}$-function of G as a function $f : V \mapsto \mathbb{N}_0$ satisfying $f(v_i) \leq r_i$ for all $i = 1, \ldots, n$. For every $v \in V$, let $f[v] = \sum_{u \in N[v]} f(u)$. An $\boldsymbol{r}$-function f is called $\boldsymbol{s}$-dominating if $f[v_i] \geq s_i$ for each $i = 1, \ldots, n$. The *weight* of an $\boldsymbol{r}$-function f is defined as $|f| = \sum_{v \in V} f(v)$. The smallest weight of an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function is called the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of G and is denoted by $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)$. As noted in [4], an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G exists if and only if $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for all $i = 1, \ldots, n$.

Note that if $\boldsymbol{r} = \boldsymbol{s} = [1, \ldots, 1]$, then the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of G is the (classical) domination number $\gamma(G)$ of G. Other special cases of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination include $\{k\}$-*domination* [2,11,20] and k-*tuple domination* [12,15,18,7] which are the cases where $\boldsymbol{r} = \boldsymbol{s} = [k, \ldots, k]$ and where $\boldsymbol{r} = [1, \ldots, 1]$ and $\boldsymbol{s} = [k, \ldots, k]$, respectively, for some $k \in \mathbb{N}$. The k-tuple domination number of a graph is denoted by $\gamma_{\times k}(G)$. The balanced case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination, where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$, was studied in [21,19] for $r, s \in \mathbb{N}$.

It is well known that the problem of computing the (classical) domination number of a graph, a special case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination, is **NP**-complete [8]. Another special case, the problem of computing the k-tuple domination number, is also

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NP-complete, even for split graphs [12]. These complexity results suggest that the problem of computing the balanced case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination may be **NP**-complete for arbitrary values of r and s.

Alon and Spencer [1] used the probabilistic method in 1991 to show that

$$\gamma(G) \leq \frac{n(1 + \ln(\delta + 1))}{\delta + 1} \tag{1}$$

for a graph G of order n with minimum degree δ. This result was, in fact, independently established earlier, without the use of the probabilistic method, by Payan [17] and Lovasz [13] in 1975. Harant et al. [10] improved this bound and showed that

$$\gamma(G) \leq n\left(1 - \delta\left(\frac{1}{\delta + 1}\right)^{1+1/\delta}\right). \tag{2}$$

Henning and Harant [9] employed the same method in 2005 to prove that

$$\gamma_{\times 2}(G) \leq n\left(\frac{\ln(1 + d) + \ln \delta + 1}{\delta}\right),$$

where $d = \frac{1}{n}\sum_{v \in V} \mathrm{d}_v$. This idea was further generalised in 2007 to k-tuple domination with the following conjecture by Rautenbach and Volkmann [18].

Conjecture 1. *If $k \in \mathbb{N}$ and G is a graph of order n with minimum degree $\delta \geq k$, then*

$$\gamma_{\times k}(G) \leq \frac{n}{\delta + 2 - k}\left(\ln(\delta + 2 - k) + \ln\left(\sum_{v \in V}\binom{\mathrm{d}_v + 1}{k - 1}\right) - \ln n + 1\right). \tag{3}$$

The special case of Conjecture 1 where $k = 3$ was established by Rautenbach and Volkmann [18] themselves, while the general conjecture was proven by Chang [3], Xu et al. [23] and Zverovich [24] independently in 2008. The results of Chang [3] and Zverovich [24] are based on the model and approach presented originally in [7], which was further developed in [5].

Rautenbach and Volkmann [18] also established another upper bound on k-tuple domination which does not depend on the degree sequence of G, namely

$$\gamma_{\times k}(G) \leq \frac{n}{\delta + 1}\left(k \ln(\delta + 1) + \sum_{i=0}^{k-1}\frac{(k - i)}{i!(\delta + 1)^{k-i-1}}\right), \tag{4}$$

where G is a graph of order n with minimum degree δ with $2k \leq (\delta + 1)/\ln(\delta + 1)$.

Recently, Gagarin et al. [6] improved the bounds (3) and (4) and proposed a bound, generalised from (1) and (2), which also does not depend on the degree sequence of G. They showed that

$$\gamma_{\times k}(G) \leq n\left(1 - \frac{\delta'}{\tilde{b}_{k-1}^{1/\delta'}(1 + \delta')^{1+1/\delta'}}\right) \tag{5}$$

for any graph G with minimum degree $\delta \geq k$, where $\delta' = \delta - k + 1$ and $\tilde{b}_t = \binom{\delta+1}{t}$.

Another probabilistic bound, again independent of the degree sequence of G, was proposed by Przybyło [16] in 2013, who showed that

$$\gamma_{\times k}(G) \leq n\left(\sum_{i=1}^{k}\frac{\ln(\delta + 2 - i) + 1}{\delta + 2 - i}\right) \tag{6}$$

for any graph G with minimum degree $\delta \geq k - 1$, where $k \leq (\delta + 2 - k)/(\ln(\delta + 2 - k) + 1)$.

After introducing three new bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number in Section 2 we present two ways to improve these bounds in Section 3. The paper closes in Section 4 with an experimental comparison of the new bounds with those in the literature, showing that none of the new bounds always outperforms the others. Furthermore, for some graphs, one of the bounds presented in Section 2 provides better results for k-tuple domination than the existing bound in (5) for certain values of k.

2. The bounds

In this section, three bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph are established for the case where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$.

Let G be a graph of order n with minimum degree δ and let $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ for some $r, s \in \mathbb{N}$ such that $r(\delta + 1) \geq s$. The function $f(v) = r$, $v \in V(G)$ is clearly an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G and therefore $\sum_{i=1}^{n} r_i = rn$ is an upper bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of G. However, this bound can always be improved since $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq \gamma_{\boldsymbol{r}'}^{\boldsymbol{s}}(G)$ where $\boldsymbol{r}' = [r - 1, \ldots, r - 1]$.