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Note

Steiner diameter of 3, 4 and 5-connected maximal planar graphs

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ABSTRACT

Let G be a connected graph of order p and S a nonempty set of vertices of G. Then the Steiner distance $d(S)$ of S is the minimum size of a connected subgraph of G whose vertex set contains S. If n is an integer, $2 \leq n \leq p$, the Steiner n-diameter, $\mathrm{diam}_n(G)$, of G is the maximum Steiner distance of any n-subset of vertices of G. This is a generalisation of the ordinary diameter, which is the case $n = 2$. We give upper bounds on the Steiner n-diameter of maximum planar graphs in terms of order and connectivity. Moreover, we construct graphs to show that the bound is asymptotically sharp. Furthermore we extend this result to 4 and 5-connected maximal planar graphs.

1. Introduction

Let G be a connected graph of order $p(G)$ and S a set of vertices of G. Let H be a connected subgraph of G of minimum size which contains S. Then H is a tree, known as a *Steiner tree* for S, and the size of H is the *Steiner distance* of S in G, denoted by $d_G(S)$. If $|S| = 2$, then the Steiner distance of S is the (ordinary) distance between the two vertices of S, so the Steiner distance generalises the ordinary distance between two vertices. Let n be an integer such that $2 \leq n \leq p$. The *n-diameter* of G, $\mathrm{diam}_n(G)$, is defined to be the maximum Steiner distance of any *n-subset* of vertices of G. The determination of a Steiner tree in a graph is a discrete analogue of the well-known geometric Steiner problem: in an Euclidean space (usually an Euclidean plane) find the shortest possible network of line segments interconnecting a set of given points. Steiner trees have application to multiprocessor computer networks. For example, it may be desired to connect a certain set of processors with a subnetwork that uses the least number of communication links. A Steiner tree for the vertices, corresponding to the processors that need to be connected, corresponds to such a desired subnetwork. Since the problem of determining the Steiner distance is known to be NP-hard [6], it is desirable to have good bounds (see for example [1,3,4]). In this paper we give upper bounds on the n-diameter in terms of order p for 3, 4 and 5-connected maximal planar graphs.

The ordinary distance between two vertices u, v of G, $d_G(u, v)$, is the length of a shortest u–v path in G. Throughout the paper we will often drop the subscript (or argument) G if no confusion can arise. The *degree*, $\deg(v)$, of a vertex v of G is the number of edges incident with it. The *minimum degree* of G, $\delta(G)$, is the smallest of the degrees of vertices in G and the *maximum degree*, $\Delta(G)$, of G is the largest of the degrees of the vertices in G. The *connectivity*, $\kappa(G)$, of G is defined as the minimum number of vertices whose deletion renders G disconnected or a trivial graph. G is *k-connected* if $\kappa(G) \geq k$. A graph is *planar* if it can be drawn in the plane with no crossing edges. A graph is *maximal planar* if it is planar, but after the addition of any edge the resulting graph is not planar.

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Distances in maximal planar graphs have been well studied in the literature. The maximum number of vertices of maximal planar graphs of given diameter and maximum degree has been determined. Hell and Seyffarth [7] have shown that the maximum number of vertices in a planar graph with diameter 2 and maximum degree $\Delta \geq 8$ is $\lfloor \frac{3}{2}\Delta + 1 \rfloor$. It was shown in [8] that maximal planar graphs of diameter 2 and maximum degree $\Delta \geq 8$ have no more than $\frac{3}{2}\Delta + 1$ vertices. It was also shown that there exist maximal planar graphs with diameter two and exactly $\lfloor \frac{3}{2}\Delta + 1 \rfloor$ vertices. Yang, Lin and Dai [9] have computed the exact maximum number of vertices in planar graphs and maximal planar graphs with diameter two and maximum degree Δ, for $\Delta < 8$.

Fulek, Morić and Pritchard [5] proved that for every connected planar graph G of order p and size m,

$$\mathrm{diam}_2(G) \leq \frac{4(p-1)-m}{3}. \tag{1}$$

Since for 3, 4 and 5-connected maximal planar graphs $m = 3p - 6$, the bound in (1) becomes

$$\mathrm{diam}_2(G) \leq \frac{p+2}{3}.$$

It is well known that for the ordinary diameter, i.e., for the case $n = 2$, if G is a k-connected graph of order p, then

$$\mathrm{diam}_2(G) \leq \left\lfloor \frac{p+k-2}{k} \right\rfloor. \tag{2}$$

The inequality (2) yields for 3-connected graphs G,

$$\mathrm{diam}_2(G) \leq \left\lfloor \frac{p+1}{3} \right\rfloor, \tag{3}$$

for 4-connected graphs G,

$$\mathrm{diam}_2(G) \leq \left\lfloor \frac{p+2}{4} \right\rfloor, \tag{4}$$

and for 5-connected graphs G,

$$\mathrm{diam}_2(G) \leq \left\lfloor \frac{p+3}{5} \right\rfloor. \tag{5}$$

So the ordinary diameters of 3, 4 and 5-connected maximal planar graphs do not exceed $\frac{p+1}{3}$, $\frac{p+2}{4}$ and $\frac{p+3}{5}$, respectively. We generalise in this paper the above bounds (3)–(5). We show that the bounds for the 2-diameter essentially also hold for the n-diameter in maximal planar graphs. In particular, we show that for maximal planar graphs G,

$$\mathrm{diam}_n(G) \leq \frac{p}{3} + \frac{8n}{3} - 5,$$

for 4-connected maximal planar graphs G,

$$\mathrm{diam}_n(G) \leq \frac{p}{4} + \frac{19n}{4} - 9,$$

and for 5-connected maximal planar graphs G,

$$\mathrm{diam}_n(G) \leq \frac{p}{5} + \frac{24n}{5} - 9.$$

We also construct graphs to show that these bounds are asymptotically sharp.

2. Results

Let G be a maximal planar graph of order p. For vertices $y, z \in V(G)$, denote by $P(z, y)$, a z–y shortest path in G. If G is rooted at a vertex, say a_0, and $i \in \mathbb{N}_0$, then the ith distance layer is the set

$$N_i := \{x \in V(G) | d_G(x, a_0) = i\}.$$

A vertex $x \in N_i$ is *active* if x has a neighbour in N_{i+1}. We denote by A_i the set of active vertices in N_i.

Lemma 2.1 is a direct consequence of a result in [2]. For completeness we give a proof.

Lemma 2.1. *Let G, and A_i be as above and let $i \in \mathbb{N}_0$.*

(i) *If G is maximal planar and $u \in A_i$, then there exist two distinct vertices in $A_i - \{u\}$ both of which are adjacent to u.*
(ii) *If G is 4-connected maximal planar and $u \in A_i$, then there exist three distinct vertices $v, w, z \in A_i - \{u\}$ such that v and w are neighbours of u and either v or w is adjacent to z.*
(iii) *If G is 5-connected maximal planar and $u \in A_i$, then there exist four distinct vertices $v, w, y, z \in A_i - \{u\}$ such that both v and w are neighbours of u, y is a neighbour of v and z is a neighbour of w.*