Discrete Applied Mathematics 179 (2014) 229–234

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

Strong edge-colouring of sparse planar graphs

Julien Bensmail [a], Ararat Harutyunyan [b], Hervé Hocquard [a], Petru Valicov [c,*]

[a] *LaBRI (Université de Bordeaux), 351 cours de la Libération, 33405 Talence Cedex, France*

[b] *Mathematical Institute, University of Oxford, United Kingdom*

[c] *École Normale Supérieure de Lyon, LIP, Équipe MC2, 46, allée d'Italie, 69342 Lyon Cedex 07, France*

ARTICLE INFO

Article history:
Received 22 January 2014
Received in revised form 25 June 2014
Accepted 20 July 2014
Available online 16 August 2014

Keywords:
Planar graphs
Girth
Proper edge-colouring
Strong edge-colouring

ABSTRACT

A strong edge-colouring of a graph is a proper edge-colouring where each colour class induces a matching. It is known that every planar graph with maximum degree Δ has a strong edge-colouring with at most $4\Delta + 4$ colours. We show that $3\Delta + 1$ colours suffice if the graph has girth 6, and 4Δ colours suffice if $\Delta \geq 7$ or the girth is at least 5. In the last part of the paper, we raise some questions related to a long-standing conjecture of Vizing on proper edge-colouring of planar graphs.

1. Introduction

A *proper edge-colouring* of a graph $G = (V, E)$ is an assignment of colours to the edges of the graph such that two adjacent edges do not use the same colour. We use the standard notation, $\chi'(G)$, to denote the chromatic index of G. A *strong edge-colouring* (called also distance 2 edge-colouring) of a graph G is a proper edge-colouring of G, such that the every set of edges using the same colour induces a matching. We denote by $\chi'_s(G)$ the *strong chromatic index* of G which is the smallest integer k such that G can be strongly edge-coloured with k colours. Strong edge-colouring has been studied extensively in the literature by different authors (see [4,5,11,6,1,8,7,2,3,9]).

The *girth* of a graph G is the length of a shortest cycle in G. We denote by Δ the maximum degree of a graph.

Perhaps the most challenging question for strong edge-colouring is the following conjecture:

Conjecture 1 (*Erdős and Nešetřil [5]*). *For every graph G, $\chi'_s(G) \leq \frac{5}{4}\Delta^2$ for Δ even and $\frac{1}{4}(5\Delta^2 - 2\Delta + 1)$ for Δ odd.*

Andersen [1] and Horák et al. [8] showed this conjecture for the case when $\Delta = 3$. When Δ is large enough, Molloy and Reed showed that $\chi'_s(G) \leq 1.998\Delta^2$ [11].

In this note, we study the strong chromatic index of planar graphs. The work in this area started with the paper of Faudree et al. [6], who proved the following theorem.

Theorem 2 (*Faudree et al. [6]*). *If G is a planar graph, then $\chi'_s(G) \leq 4\Delta + 4$, for $\Delta \geq 3$.*

The proof of Theorem 2 uses the Four Colour Theorem. The authors also provided a construction of planar graphs of girth 4 which satisfies $\chi'_s(G) = 4\Delta - 4$. Hence, the bound of Theorem 2 is optimal up to an additive constant.

The same authors also conjectured that for $\Delta = 3$ the bound can be improved.

* Corresponding author.
E-mail address: petru.valicov@ens-lyon.fr (P. Valicov).

http://dx.doi.org/10.1016/j.dam.2014.07.006

Conjecture 3 (*Faudree et al. [6]*). *If G is a planar subcubic graph then $\chi'_s(G) \leq 9$.*

Hocquard et al. obtained the following weakening of Conjecture 3.

Theorem 4 (*Hocquard et al. [7]*). *If G is a planar graph with $\Delta \leq 3$ containing neither induced 4-cycles, nor induced 5-cycles, then $\chi'_s(G) \leq 9$.*

An interesting question is to see how the strong chromatic index behaves for sparse planar graphs. For instance, when the girth is large enough the strong chromatic index decreases to the near optimal lower bound, as showed in the following theorems:

Theorem 5 (*Borodin and Ivanova [2]*). *If G is a planar graph with maximum degree $\Delta \geq 3$ and girth $g \geq 40\lfloor \frac{\Delta}{2} \rfloor$, then $\chi'_s(G) \leq 2\Delta - 1$.*

Recently this result was improved for $\Delta \geq 6$:

Theorem 6 (*Chang et al. [3]*). *If G is a planar graph with maximum degree $\Delta \geq 4$ and girth $g \geq 10\Delta + 46$, then $\chi'_s(G) \leq 2\Delta - 1$.*

For smaller values of the girth, Hudák et al. [9] improved the bound in Theorem 2.

Theorem 7 (*Hudák et al. [9]*). *If G is a planar graph with girth $g \geq 6$, then $\chi'_s(G) \leq 3\Delta + 6$.*

Our main result in this paper improves the upper bound in Theorem 7. In particular, we show the following.

Theorem 8. *If G is a planar graph with girth $g \geq 6$, then $\chi'_s(G) \leq 3\Delta + 1$.*

Moreover, in Section 3, by a more careful analysis of the proof of Theorem 2 given in [6] and by using some results on proper edge-colouring, we obtain the following strengthening.

Theorem 9. *Let G be a planar graph with maximum degree Δ and girth g. If G satisfies one of the following conditions below, then $\chi'_s(G) \leq 4\Delta$*

- $\Delta \geq 7$,
- $\Delta \geq 5$ *and* $g \geq 4$,
- $g \geq 5$.

Before proving our results we introduce some notation.

Notation. Let G be a graph. Let $d(v)$ denote the degree of a vertex v in G. A vertex of degree k is called a k-vertex. A k^+-vertex (respectively, k^--vertex) is a vertex of degree at least k (respectively, at most k). A k_l-vertex is a k-vertex adjacent to exactly l 2-vertices. A *bad 2-vertex* is a 2-vertex adjacent to another 2-vertex. When speaking about a vertex as a neighbour, same notations apply just by replacing the word "vertex" with "neighbour". Two edges are at distance 1 if they share one of their ends and they are at distance 2 if they are not at distance 1 and there exists an edge adjacent to both of them. We define $N_2[uv]$ as the set of edges at distance at most 2 from the edge uv and $N_2(uv) = N_2[uv] - uv$. Given an edge-colouring of G, we denote by $SC(N_2(uv))$ ($SC(N_2[uv])$ respectively) the set of colours used by edges in $N_2(uv)$ ($N_2[uv]$ respectively). We denote by $N(v)$ the neighbourhood of the vertex v, *i.e.*, the set of its adjacent vertices. Finally, we use $[\![n]\!]$ to denote the set of integers $\{1, 2, \ldots, n\}$.

2. Proof of Theorem 8

2.1. Structural properties

We proceed by contradiction. Let H be a counterexample to the theorem that minimizes $|E(H)| + |V(H)|$. By minimality of H we can assume that it is connected and that by Theorem 4 it has $\Delta(H) \geq 4$.

Claim 1. *H satisfies the following properties:*

1. *H does not contain a 1-vertex adjacent to a 4^--vertex.*
2. *H does not contain a 2-vertex adjacent to two 3^--vertices.*
3. *H does not contain a 2-vertex adjacent to a 3^--vertex and either a 4_2-vertex or a 4_3-vertex.*
4. *H does not contain a 2-vertex adjacent to a 4_3-vertex and to a 4_2-vertex.*
5. *If $k \geq 4$, then H does not contain a k-vertex adjacent to $k-2$ 1-vertices; if the k-vertex is adjacent to $k-3$ 1-vertices, then it has no other 2^--neighbour.*
6. *If $k \geq 4$, then H does not contain a k-vertex adjacent to k 2^--vertices.*
7. *If $k \geq 5$, then H does not contain a k-vertex u with $N(u) = \{u_1, u_2, \ldots, u_{k-1}, x\}$, such that each u_i with $i \in [\![k-1]\!]$ is a 2^--vertex and u_1 is either a 1-vertex or a 2-vertex adjacent to either a 3^--vertex or a 4_3-vertex.*
8. *If $k \geq 5$, then H does not contain a k-vertex adjacent to $k-2$ vertices of degree 2, $u_1, \ldots, u_{k-2}$, such that for $i \in [\![k-3]\!]$, each u_i is adjacent to either a 3^--vertex or a 4_3-vertex.*
9. *If $k \geq 5$ and $1 \leq \alpha \leq k-4$, then H does not contain a k-vertex adjacent to α 1-vertices and to $k-2-\alpha$ vertices of degree 2, $u_1, \ldots, u_{k-2-\alpha}$, such that for $i \in [\![k-3-\alpha]\!]$ each u_i is adjacent to either a 3^--vertex or a 4_3-vertex.*