



The majority game with an arbitrary majority



John R. Britnell^{a,*}, Mark Wildon^b

^a Department of Mathematics, Imperial College London, South Kensington Campus, London SW7 2AZ, United Kingdom

^b Department of Mathematics, Royal Holloway, University of London, Egham, Surrey TW20 0EX, United Kingdom

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ABSTRACT

The k -majority game is played with n numbered balls, each coloured with one of two colours. It is given that there are at least k balls of the majority colour, where k is a fixed integer greater than $n/2$. On each turn the player selects two balls to compare, and it is revealed whether they are of the same colour; the player's aim is to determine a ball of the majority colour. It has been correctly stated by Aigner that the minimum number of comparisons necessary to guarantee success is $2(n - k) - B(n - k)$, where $B(m)$ is the number of 1s in the binary expansion of m . However his proof contains an error. We give an alternative proof of this result, which generalizes an argument of Saks and Werman.

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1. Introduction

Fix n and $k \in \mathbf{N}$ with $k > n/2$. The k -majority game is played with n numbered balls which are each coloured with one of two colours. It is given that there are at least k balls of the majority colour. On each turn the player selects two balls to compare, and it is revealed whether they are of the same colour, or of different colours. The player's objective is to determine a ball of the majority colour. We write $K(n, k)$ for the minimum number of comparisons that will guarantee success. We write $B(m)$ for the number of digits 1 in the binary representation of $m \in \mathbf{N}_0$. The object of this paper is to prove the following theorem.

Theorem 1. *If $n, k \in \mathbf{N}$ and $k \leq n$ then $K(n, k) = 2(n - k) - B(n - k)$.*

This theorem has been stated previously, as Theorem 3 of [1, page 14]. We believe, however, that there is a flaw in the proof offered there of the lower bound for $K(n, k)$, i.e. the fact that $2(n - k) - B(n - k)$ comparisons are necessary. The error arises in Case (ii) of the proof of Lemma 1, in which it is implicitly assumed that if it is optimal at some point for the player to compare balls i and t , then there exist two balls j and ℓ which it is optimal to compare on the next turn, irrespective of the answer received when balls i and t are compared. The proof of Theorem 3 requires an analogue of Lemma 1, stated as Lemma 3, which inherits the same error. The argument for Lemma 1 of [1] is based on Lemma 5.1 in [9], which contains the same flaw; the authors are grateful to Prof. Aigner and Prof. Wiener for confirming these errors.¹

In [8], Saks and Werman have shown that $K(2m + 1, m + 1) = 2m - B(m)$. (An independent proof, using an elegant argument on the game tree, was later given by Alonso, Reingold and Schott [3].) The original contribution of this paper is to supply a correct proof that $2(n - k) - B(n - k)$ questions are necessary in the general case, by generalizing the argument of Saks and Werman [8].

* Corresponding author.

E-mail addresses: j.britnell@imperial.ac.uk (J.R. Britnell), mark.wildon@rhul.ac.uk (M. Wildon).

¹ Personal communications.

We remark that an alternative setting for the majority problem replaces the n balls with a room of n people. Each person is either a knight, who always tells the truth, or a knave, who always lies. The question ‘Person i , is person j a knight?’ corresponds to a comparison between balls i and j . (The asymmetry in the form of the questions is therefore illusory.) In [1, Theorem 6], Aigner gives a clever questioning strategy which demonstrates that $2(n - k) - B(n - k)$ questions suffice, even when knaves are replaced by spies (Aigner’s unreliable people), who may answer as they see fit. He subsequently uses his Lemma 3 to show that $2(n - k) - B(n - k)$ questions are also necessary; Theorem 1 can be used to replace this lemma, and so repair the gap in the proof of Theorem 6 of [1].

We refer the reader to [1] and the recent preprint [4] for a number of results on further questions that arise in this setting. Many variants of the problem have been studied, for instance involving balls of more than two colours (see [2]), or with comparisons involving more than two balls (see [7]).

2. Preliminary reformulation

We begin with a standard reformulation of the problem that follows [1, §2]. In the special case $n = 2m + 1$ and $k = m + 1$, it may also be found in [8, §4] and [9, §2, §3]. A position in a k -majority game corresponds to a graph on n vertices, in which there is an edge, labelled either ‘same’ or ‘different’, between vertices i and j if balls i and j have been compared. Each connected component of this graph admits a unique bipartition into parts corresponding to balls of the same colour. If C is a component with bipartition $\{X, Y\}$ where $|X| \geq |Y|$ then we define the *weight* of C to be $|X| - |Y|$.

The weights of the distinct components of the graph contain all of the essential information about the position, and we may therefore reformulate the k -majority game as a two player adversarial game played on multisets of non-negative integers. We shall call these multisets *positions* and their elements *weights*. The players will be known as the *Selector* and the *Assigner*. The starting position is the multiset $\{1, \dots, 1\}$ containing n elements, corresponding to the n trivial connected components of the null graph on n vertices.

In each turn, the Selector selects two distinct multiset elements w and w' , with $w \geq w'$, and the Assigner chooses to replace them with either $w + w'$ or $w - w'$. This corresponds to the situation in the original game when a ball from a component C of weight w is compared to a ball from a different component C' of weight w' ; the components C and C' become connected by an edge, and the new component thus formed has weight either $w + w'$ or $w - w'$ depending on the result of the comparison. Since comparisons of balls in the same component give no information, we need only consider comparisons of this type.

We require a victory condition for the Selector. Let $e = k - (n - k)$ be the minimum possible excess of the majority colour over the minority colour, and let $w_1, \dots, w_c$ be the weights in a given position. (We shall always assume that the weights are listed in non-increasing order, so that $w_i \geq w_{i+1}$.) We remark that $\sum_i w_i \equiv n \pmod{2}$, since this is true of the initial position, and since the parity of the sum of weights is preserved at each turn. Hence $w_1 + \dots + w_c = 2s + e$ for some $s \in \mathbb{N}_0$.

Suppose that the balls in the larger part of the component of (largest) weight w_1 are of the minority colour. Then we see that $w_1 \leq s$, since we must have $-w_1 + w_2 + \dots + w_c \geq e$. (This observation is equivalent to equation (14) in [1].) It follows that the Selector wins as soon as a position $M = \{w_1, \dots, w_c\}$ is reached such that

$$w_1 \geq s + 1,$$

where s is determined by $2s + e = w_1 + \dots + w_c$, since in this situation the balls in the larger part of the largest weight component are known to have the majority colour. Following [8], we say that such a position M is *final*.

Our concern is with the number of turns required for victory. We observe that the cardinality of the multiset is reduced by 1 at each turn, and so if the position is $M = \{w_1, \dots, w_c\}$, then the number of turns that have elapsed is $n - c$. We define the *value* of a general position M to be the number of elements in a final position reached from M , assuming, as ever, optimal play by both sides. We denote the value of M by $V(M)$.

The result we require, that $2(n - k) - B(n - k)$ questions are necessary to identify a ball of the majority colour in the k -majority game, is equivalent to the following proposition.

Proposition 2. *Let $n \in \mathbb{N}$ and let $k > n/2$. The value of the starting position in the k -majority game is at most $B(n - k) + k - (n - k)$.*

3. Generalized Saks–Werman statistics

If M is a position in a majority game and N is a submultiset of M then we shall say that N is a *subposition* of M . Let $\bar{N}$ denote the complement of N in M and let $\|M\|$ denote the sum of all the elements of M . Let $\varepsilon_M(N) = \|\bar{N}\| - \|N\|$. For $e \in \mathbb{N}$ and a position M such that $\|M\|$ and e have the same parity, we define

$$\delta_e(M) = \sum_{\substack{N \\ \varepsilon_M(N) \geq e}} (-1)^{\|N\|}.$$

Thus a subposition of M contributes to $\delta_e(M)$ if and only if it corresponds to a colouring of the balls in which the excess of the majority colour over the minority colour is at least e . We note that when $e = 1$ we have $\delta_1(M) = -f_M(-1)$ where f_M is the

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