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Game total domination for cycles and paths

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ABSTRACT

In this paper, we continue the study of the recently introduced total domination game in graphs. A vertex u in a graph G totally dominates a vertex v if u is adjacent to v in G. A total dominating set of G is a set S of vertices of G such that every vertex of G is totally dominated by a vertex in S. The total domination game played on G consists of two players, named Dominator and Staller, who alternately take turns choosing vertices of G such that each chosen vertex totally dominates at least one vertex not totally dominated by the vertices previously chosen. Dominator's goal is to totally dominate the graph as fast as possible, and Staller wishes to delay the process as much as possible. The game total domination number, $\gamma_{tg}(G)$, of G is the number of vertices chosen when Dominator starts the game and both players play optimally. The Staller-start game total domination number, $\gamma'_{tg}(G)$, of G is the number of vertices chosen when Staller starts the game and both players play optimally. In this paper we determine $\gamma_{tg}(G)$ and $\gamma'_{tg}(G)$ when G is a cycle or a path. In particular, we show that for a cycle C_n on $n \geq 3$ vertices, $\gamma_{tg}(C_n) = \lfloor \frac{2n+1}{3} \rfloor - 1$ when $n \equiv 4 \pmod 6$ and $\gamma_{tg}(C_n) = \lfloor \frac{2n+1}{3} \rfloor$ otherwise. Further, $\gamma'_{tg}(C_n) = \lfloor \frac{2n}{3} \rfloor - 1$ when $n \equiv 2 \pmod 6$ and $\gamma'_{tg}(C_n) = \lfloor \frac{2n}{3} \rfloor$ otherwise. For a path P_n on $n \geq 3$ vertices, we show that $\gamma'_{tg}(P_n) = \lceil \frac{2n}{3} \rceil$. Further, $\gamma_{tg}(P_n) = \lfloor \frac{2n}{3} \rfloor$ when $n \equiv 5 \pmod 6$ and $\lceil \frac{2n}{3} \rceil$ otherwise.

1. Introduction

In this paper, we continue the study of the domination game in graphs introduced by Brešar, Klavžar, and Rall [3] and extensively studied afterward in [1,4,2,5–8,10,13,14,16,17] and elsewhere.

The total version of the domination game was recently investigated in [11], where it was shown, among other results, that the two versions differ significantly. A vertex u in a graph G *totally dominates* a vertex v if u is adjacent to v in G. A *total dominating set* of G is a set S of vertices of G such that every vertex of G is totally dominated by a vertex in S. The *total domination game*, played on a graph G consists of two players called *Dominator* and *Staller* who take turns choosing a vertex from G. Each chosen vertex must totally dominate at least one vertex not totally dominated by the set of vertices previously chosen. Suppose that at a particular point in the game some subset C of vertices has been chosen by the players, and it is the turn of one of the two players. We say that a vertex v of G is *playable* if v is adjacent to a vertex that is not totally dominated by C. If the player chooses v, then we say the player *played* v and we refer to this choice as the *move* of that player. For emphasis we may also say it was a *legal* move. A vertex is *unplayable* if it is not a legal move. The game ends when G has no playable

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vertices in which case the set of vertices chosen is necessarily a total dominating set in G. Dominator wishes to end the game with a minimum number of vertices chosen, and Staller wishes to end the game with as many vertices chosen as possible.

The *game total domination number*, $\gamma_{\rm tg}(G)$, is the number of vertices chosen when Dominator starts the game and both players play optimally. The *Staller-start game total domination number*, $\gamma'_{\rm tg}(G)$, is the number of vertices chosen when Staller starts the game and both players play optimally. For notational simplicity, we shall simply refer to the Dominator-start total domination game and the Staller-start total domination game as the *Dominator-start game* and the *Staller-start game*, respectively.

Determining the exact value of $\gamma_{\rm tg}(G)$ and $\gamma'_{\rm tg}(G)$ for special classes of graphs G is a challenging problem. Our aim in this paper is to determine the game total domination number and the Staller-start game total domination number of a cycle and a path.

1.1. Notation and graph theory terminology

For notation and graph theory terminology, we in general follow [14]. We denote the *degree* of v in a graph G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from the context. If $d_G(v) = 1$, we call v a *leaf* in G. The *open neighborhood* of a vertex v is the set $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. A *cycle* and *path* on n vertices are denoted by C_n and P_n, respectively. If X and Y are subsets of vertices in a graph G, then the set X *totally dominates* the set Y in G if every vertex of Y is adjacent to at least one vertex of X. In particular, if X totally dominates the vertex set $V(G)$, then X is a total dominating set in G. Total domination in graphs is very well studied in graph theory. The literature on this subject has been surveyed and detailed in a recent book on this topic that can be found in [15]. A survey of total domination in graphs can be found in [9]. We use the notation $[k] = \{1, 2, \ldots, k\}$.

1.2. Known results

A *partially total dominated graph* is a graph together with a declaration that some vertices are already totally dominated; that is, they need not be totally dominated in the rest of the game. If G is a graph and $A \subseteq V(G)$ is such a set, we will denote with $G|A$ this partially total dominated graph. Moreover, $\gamma_{\rm tg}(G|A)$ and $\gamma'_{\rm tg}(G|A)$ are the minimum number of moves needed to finish the game on $G|A$ when Dominator or Staller starts, respectively. In [11], the authors present a key lemma, named the *Total Continuation Principle*.

Lemma 1 ([11]). *(Total Continuation Principle) Let G be a graph, $A, B \subseteq V(G)$, and let $G|A$ and $G|B$ be the corresponding partially total dominated graphs. If $B \subseteq A$, then $\gamma_{\rm tg}(G|A) \le \gamma_{\rm tg}(G|B)$ and $\gamma'_{\rm tg}(G|A) \le \gamma'_{\rm tg}(G|B)$.*

The Total Continuation Principle implies, in particular, that the number of moves in the Dominator-start game and the Staller-start game when played optimally can differ by at most 1.

Theorem 2 ([11]). *For every graph G, we have $|\gamma_{\rm tg}(G) - \gamma'_{\rm tg}(G)| \le 1$.*

A bound on the game total domination number for general graphs is established in [12] where it is shown that if G is a graph on n vertices in which every component contains at least three vertices, then $\gamma_{\rm tg}(G) \le \frac{4}{5}n$ and $\gamma'_{\rm tg}(G) \le \frac{1}{5}(4n+2)$.

2. Cycles

In this section we study the game total domination number, $\gamma_{\rm tg}(G)$, and the Staller-start game total domination number, $\gamma'_{\rm tg}(G)$, when G is a cycle C_n on n vertices. We prove that $\gamma_{\rm tg}(C_n)$ and $\gamma'_{\rm tg}(C_n)$ take on the values $\tau(n)$ and $\tau'(n)$, respectively, where

$$\tau(n) = \begin{cases} \left\lfloor \dfrac{2n+1}{3} \right\rfloor - 1 & \text{when } n \equiv 4 \pmod 6 \\ \left\lfloor \dfrac{2n+1}{3} \right\rfloor & \text{otherwise} \end{cases}$$

and

$$\tau'(n) = \begin{cases} \left\lfloor \dfrac{2n}{3} \right\rfloor - 1 & \text{when } n \equiv 2 \pmod 6 \\ \left\lfloor \dfrac{2n}{3} \right\rfloor & \text{otherwise.} \end{cases}$$

We shall therefore prove the following result in this section.

Theorem 3. *For all $n \ge 3$, we have $\gamma_{\rm tg}(C_n) = \tau(n)$ and $\gamma'_{\rm tg}(C_n) = \tau'(n)$.*

In order to prove Theorem 3 we prove a series of lemmas that establish upper and lower bounds on $\gamma_{\rm tg}(C_n)$ and $\gamma'_{\rm tg}(C_n)$. For small n, the values of $\gamma_{\rm tg}(C_n)$ and $\gamma'_{\rm tg}(C_n)$ are easy to compute (or can be checked by computer) and are shown in Table 1.