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Strong equality of Roman and weak Roman domination in trees

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ABSTRACT

We provide a constructive characterization of the trees for which the Roman domination number strongly equals the weak Roman domination number, that is, for which every weak Roman dominating function of minimum weight is a Roman dominating function. Our characterization is based on five simple extension operations, and reveals several structural properties of these trees.

1. Introduction

We consider finite, simple, and undirected graphs, and use standard terminology and notation.

Let G be a graph, and let X be a subset of the vertex set $V(G)$ of G. For a function $f : V(G) \to \mathbb{R}$, let $f(X) = \sum_{u\in X} f(u)$, and let the *weight* of f be $f(V(G))$. Furthermore, if u and v are distinct vertices of G, then let

$$f_{v\to u} : V(G) \to \mathbb{R} : x \mapsto \begin{cases} f(u) + 1, & x = u, \\ f(v) - 1, & x = v, \quad \text{and} \\ f(x), & x \in V(G) \setminus \{u, v\}. \end{cases}$$

A set D of vertices of G is X-*dominating* if every vertex in $X \setminus D$ has a neighbor in D. For a positive integer k, let $[k] = \{i \in \mathbb{N} : i \le k\}$.

Roman domination and weak Roman domination were introduced in [11] and [9], respectively. For our current purposes, we introduce slightly more general notions. A *Roman dominating function for* (G, X), a (G, X)-*RDF* for short, is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex u in X with $f(u) = 0$ has a neighbor v with $f(v) = 2$. The *Roman domination number* $\gamma_R(G, X)$ of (G, X) is the minimum weight of a (G, X)-RDF, and a (G, X)-RDF of weight $\gamma_R(G, X)$ is *minimum*. The *Roman domination number* $\gamma_R(G)$ of G is $\gamma_R(G, V(G))$. A *weak Roman dominating function for* (G, X), a (G, X)-*WRDF* for short, is a function $g : V(G) \to \{0, 1, 2\}$ such that every vertex u in X with $g(u) = 0$ has a neighbor v with $g(v) \ge 1$ such that the set $\{x \in V(G) : g_{v\to u}(x) \ge 1\}$ is X-dominating. The *weak Roman domination number* $\gamma_r(G, X)$ of (G, X) is the minimum weight of a (G, X)-WRDF, and a (G, X)-WRDF of weight $\gamma_r(G, X)$ is *minimum*. The *weak Roman domination number* $\gamma_r(G)$ of G is $\gamma_r(G, V(G))$.

Since every (G, X)-RDF is also a (G, X)-WRDF, we have $\gamma_r(G, X) \le \gamma_R(G, X)$, and, in particular,

$$\gamma_r(G) \le \gamma_R(G). \tag{1}$$

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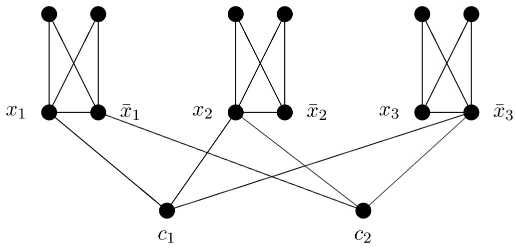

Fig. 1. The graph G for the two clauses $C_1 = x_1 \vee x_2 \vee \bar{x}_3$ and $C_2 = \bar{x}_1 \vee x_2 \vee \bar{x}_3$ over the three boolean variables x_1, x_2, and x_3.

The motivation for the current work was a problem posed by Chellali et al. [1] who asked for a characterization of the trees that satisfy (1) with equality (cf. Problem 15 in [1]). In view of the following result, the extremal graphs for (1) do most likely not have a good characterization in general, which justifies the restriction to trees.

Theorem 1. *For a given graph G, it is NP-hard to decide whether* $\gamma_r(G) = \gamma_R(G)$.

Proof. We describe a polynomial reduction from 3Sat. Therefore, let F be a 3Sat instance with clauses $C_1, \ldots, C_m$ over the boolean variables $x_1, \ldots, x_n$. We construct a graph G whose order is polynomially bounded in terms of n and m such that F is satisfiable if and only if $\gamma_r(G) = \gamma_R(G)$. Therefore, for every boolean variable x_i, create a copy $G(x_i)$ of $K_4 - e$ and denote the two vertices of degree 3 in $G(x_i)$ by x_i and $\bar{x}_i$. For every clause C_j, create a vertex c_j. For every literal $x \in \{x_i, \bar{x}_i\}$ and every clause C_j such that x appears in C_j, connect the vertex denoted x in $G(x_i)$ with c_j by an edge. See Fig. 1 for an example of the construction.

Clearly, for every $(G, V(G))$-WRDF g and every $i \in [n]$, we have $g(V(G(x_i))) \geq 2$, which implies $2n \leq \gamma_r(G) \leq \gamma_R(G)$. Since

$$g : V(G) \to \{0, 1, 2\} : x \mapsto \begin{cases} 1, & x \in \{x_i : i \in [n]\} \cup \{\bar{x}_i : i \in [n]\}, \quad \text{and} \\ 0, & x \in V(G) \setminus (\{x_i : i \in [n]\} \cup \{\bar{x}_i : i \in [n]\}) \end{cases}$$

is a $(G, V(G))$-WRDF, we have $\gamma_r(G) = 2n$. Furthermore, $\gamma_R(G) = 2n$ holds if and only if there is a $(G, V(G))$-RDF f such that for every $i \in [n]$, f assigns the value 2 to either x_i or $\bar{x}_i$, and to every other vertex, f assigns the value 0. Since such a $(G, V(G))$-RDF indicates a satisfying truth assignment for F, and, conversely, a satisfying truth assignment for F leads to such a $(G, V(G))$-RDF, we obtain that $\gamma_r(G) = \gamma_R(G)$ if and only if F is satisfiable. □

A typical solution for the problem posed by Chellali et al. [1] would be a so-called *constructive characterization*, that is, a recursive constructive description of the set of all extremal trees for (1). There are many examples of such characterizations in the literature [3–5,8]. Usually, they involve some few small extremal trees together with a small set of simple extension operations that are applied recursively in order to create all larger extremal trees. Sometimes additional information, such as certain labels or suitable subsets, has to be maintained in order to apply the extension operations properly. The Roman domination number as well as the weak Roman domination number of a given tree can be calculated by simple linear time algorithms based on standard approaches [9]. This implies that the extremal trees for (1) can easily be recognized in linear time, and a constructive characterization of these trees would only be beneficial if it reveals interesting structural properties and/or is considerably simpler than the two linear time algorithms. We did not arrive at a completely satisfactory solution of the problem posed by Chellali et al. [1], because all our constructive characterizations were essentially equivalent to implicit executions of the two linear time algorithms. Therefore, we turn to a variation of the posed problem based on the concept of *strong equality*, which was first introduced by Haynes and Slater in [7].

The Roman domination number of a graph G *strongly equals* the weak Roman domination number of G if every minimum $(G, V(G))$-WRDF is a $(G, V(G))$-RDF. Since the Roman domination number of G equals the weak Roman domination number of G if some – and not necessarily all – minimum $(G, V(G))$-WRDF is a $(G, V(G))$-RDF, strong equality implies equality. Our main result presented in the next section is a constructive characterization, based on five simple extension operations, of the trees for which the Roman domination number strongly equals the weak Roman domination number. Further examples of characterizations of strong equalities can be found in [2,6,10]. In a concluding section we discuss a possible constructive characterization of the extremal trees for (1) and its weaknesses.

2. Constructive characterization of strong equality

Instead of just trees our construction involves slightly more general objects, which are trees together with two suitable vertex subsets. Therefore, let $\mathcal{S}$ be the set of all triples (T, X, Y) with the following properties:

- T is a tree, and X and Y are sets of vertices of T.
- Every minimum (T, X)-WRDF is a (T, X)-RDF.
- Y is the set of all vertices u of T for which there is some minimum (T, X)-WRDF g such that
  - either $g(u) \geq 1$,
  - or $g(u) = 0$ and u has a neighbor v with $g(v) \geq 1$ such that the set $\{x \in V(T) : g_{v \to u}(x) \geq 1\}$ is X-dominating.