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A polyhedral investigation of star colorings

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ABSTRACT

Given a weighted undirected graph G and a nonnegative integer k, the maximum k-star colorable subgraph problem consists of finding an induced subgraph of G which has maximum weight and can be star colored with at most k colors; a star coloring does not color adjacent nodes with the same color and avoids coloring any 4-path with exactly two colors. In this article, we investigate the polyhedral properties of this problem. In particular, we characterize cases in which the inequalities that appear in a natural integer programming formulation define facets. Moreover, we identify graph classes for which these base inequalities give a complete linear description. We then study path graphs in more detail and provide a complete linear description for an alternative polytope for $k = 2$. Finally, we derive complete balanced bipartite subgraph inequalities and present some computational results.

1. Introduction

For an undirected graph $G = (V, E)$, a coloring of G is an assignment of colors to the nodes of G such that no two adjacent nodes receive the same color. A *star coloring* of G is a coloring such that no four consecutive nodes on a path are colored with exactly two colors. Star colorings were introduced by Grünbaum [15] and have applications in the computation of sparse Hessians, see, e.g., Coleman and Moré [7] and Section 2 for more details.

The goal of this paper is to investigate the polyhedral properties of star colorings; to the best of our knowledge this has not been considered before. We consider a positive integer k and the corresponding polytope $\mathrm{P}_k^\star(G)$, which is the convex hull of incidence vectors of star colorings of subgraphs of G using at most k colors.

An integer programming (IP) formulation of star colorings, with binary variables indicating whether a node receives a given color, contains three nontrivial families of inequalities: *packing inequalities*, indicating that a node receives at most one color, *edge inequalities* forbidding adjacent nodes with the same color, and *star inequalities* ensuring the star condition mentioned above. We show that packing inequalities always define facets if $k \geq 2$. Edge inequalities are (possibly) dominated by clique inequalities, which define facets if and only if the cliques are maximal. This is similar to the stable set polytope, see, e.g., Chvátal [5] and Padberg [30]. A characterization for star inequalities to define facets is more complex, depending on the structure of the neighborhood of the path.

Afterwards, we focus on complete linear descriptions of $\mathrm{P}_k^\star(G)$. We first characterize graphs for which the above three families of inequalities and nonnegativity constraints completely describe the polytope. It turns out that the star inequalities for the corresponding graphs are in fact redundant. This highlights the surprising fact that the mentioned inequalities never suffice to describe the polytope for cases in which the star coloring constraints are "active" (non-redundant).

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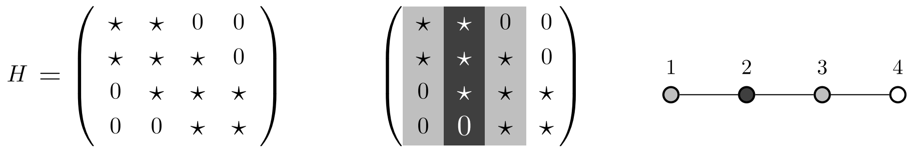

(a) Nonzero pattern of Hessian. (b) Column partition. (c) Adjacency graph.

Fig. 1. Example for the computation of sparse Hessians.

A base case for which the investigation of the structure of the polytope $\mathrm{P}^\star_k(G)$ seems to be particularly interesting are path graphs. However, it turns out that $\mathrm{P}^\star_k(G)$ is rather complicated for $k = 2$ even for path graphs. We therefore consider a projected formulation that indicates whether a node is colored or not, but does not distinguish the two colors. We then obtain a complete description using only trivial inequalities plus "projected star inequalities"; in fact, we prove that this description is totally dual integral (TDI). The projection, however, is only valid for $k = 2$.

Furthermore, we investigate a generalization of star inequalities to complete bipartite graphs. We prove that the corresponding inequalities define facets of $\mathrm{P}^\star_k(G)$, if G is a balanced complete bipartite graph. These inequalities are then used in an implementation, for which we present computational experiments.

As mentioned above, one application of the star coloring problem is the efficient computation of sparse Hessians via automatic differentiation, see, e.g., Coleman and Moré [7], Gebremedhin et al. [12–14]. However, star colorings have been investigated in the literature mainly with respect to lower and upper bounds, i.e., the computation of star colorings with the least number of used colors (*star chromatic number*) and lower bounds on the number of colors needed, see, e.g., Albertson et al. [2], Fertin et al. [9], and Lyons [25]. Lyons [25] identifies graph classes for which the number of colors needed coincides with the number of colors in an ordinary coloring. Moreover, Lyons [26] investigates classes of graphs for which the star chromatic number equals the so-called acyclic chromatic number (acyclic colorings also arise in the context of sparse Hessians).

The polyhedral properties of ordinary colorings have been investigated, for instance, in Mendez-Díaz and Zabala [28,27] and Coll et al. [8]. The problem of finding subgraphs of a given graph that can be colored with a specific number of colors is studied by Narasimhan [29]. A polyhedral analysis of this problem and an investigation of its algorithmic treatment as well as the handling of color and graph symmetries can be found in [19,20].

The structure of this paper is as follows: In Section 2 we describe the application of star colorings in the computation of sparse Hessians, and in Section 3 we introduce some basic notation. A polyhedral description for the maximum k-star colorable subgraph problem is developed in Section 4, and in Section 5 we provide graph classes for that we can find a complete linear description. In Section 6 we derive a projected polyhedral model for $k = 2$, and we study polyhedral properties of the new model and connections to the previously defined polytope. Section 7 applies the results from the previous section to path graphs, whereas Section 8 provides further facet inducing inequalities for $\mathrm{P}^\star_k(G)$. Computational results on the maximum k-star colorable subgraph problem are presented in Section 9, and we conclude this paper in Section 10.

2. Computation of sparse Hessians

Since the above mentioned application to the computation of sparse Hessians is important, we will briefly describe the connection to star colorings in this section. The presentation is based on Gebremedhin et al. [12, Section 4].

Suppose we are given a twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ and the goal is to efficiently compute its Hessian H, e.g., in a (globalized) Newton or SQP method (see, e.g., Bonnans et al. [4]). Consider, for example, a Hessian with the sparsity pattern as shown in Fig. 1(a). The first two components of the directional derivative of ∇f along $e_1 + e_4$ equal the nonzeros of the first column of H, and its last two components are the nonzero entries in the fourth column of H; here, e_i is the ith unit vector. This is true since the nonzeros of the first and fourth column are disjoint. The final Hessian can then be completed by taking the derivative of ∇f along e_2 and e_3, respectively. In contrast, the naïve approach would require four derivatives.

This argument can be easily extended to the general case by partitioning the columns such that for each row there is at most one column in each part which has a nonzero in this row. Then for each part π, the components of the derivative of ∇f along the sum of unit vectors for each index in π give the nonzeros of the corresponding columns in H. The efficiency of this scheme increases if the number of parts decreases. Furthermore, we can obtain a speed-up by exploiting the symmetry of H, since it suffices to find a partition of the columns of H such that we are able to recover H_{ij} or H_{ji}.

Coleman and Moré [7] show that a column partition of H (assuming that the diagonal entries are nonzero) allow such a symmetric reconstruction if and only if it induces a star coloring of the adjacency graph of H. The *adjacency graph* is defined by introducing a node for each column of H; there is an edge between two distinct nodes i and j if and only if $H_{ij} = H_{ji} \neq 0$.

Consider again the example in Fig. 1. Taking the derivative d of ∇f along $e_1 + e_3$ allows to compute H_{11}, H_{33}, and H_{43}, but only $H_{21} + H_{23}$. If we take the derivative along $e_2 + e_4$, we cannot recover H_{32} and H_{34}. Thus, taking just two derivatives (parts) does not suffice to compute H, since columns $\{1, 3\}$ would receive one color and $\{2, 4\}$ a second one. Hence, the