Discrete Applied Mathematics 208 (2016) 79–87

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On the skew-permanental polynomials of orientation graphs ☆

Wei Li

Department of Applied Mathematics, School of Science, Northwestern Polytechnical University, Xi'an, Shaanxi, 710129, PR China

ARTICLE INFO

Article history:
Received 14 January 2016
Accepted 29 March 2016
Available online 23 April 2016

Keywords:
Skew-permanental polynomial
Characteristic polynomial
Orientation graph

ABSTRACT

Let $\pi(G,x)=\sum_{i=0}^{n} b_i x^{n-i}$ be the permanental polynomial of a graph G, and $\pi_s(G^\sigma,x)=\sum_{i=0}^{n} d_i x^{n-i}$ the skew-permanental polynomial of an orientation graph G^σ. In this paper, we investigate first the orientation graph G^σ of a bipartite graph G satisfying $|d_n|=|b_n|$. Furthermore, we characterize the orientation of a bipartite graph with $|d_i|=|b_i|$ for each i. Then we establish the recursion formulas for the skew-permanental polynomials of orientation graphs. In addition, we characterize the graphs whose skew-permanental polynomials can be expressed by the corresponding characteristic polynomials.

1. Introduction

Throughout this paper, G denotes a simple finite undirected graph. We use $n(G)$ (or simply n) to denote the number of vertices of G. Let $A(G)$ be the adjacency matrix of G. The *characteristic polynomial* of G is

$$\phi(G,x)=\det(xI-A(G))=\sum_{i=0}^{n} a_i x^{n-i},$$

where I is the $n\times n$ identity matrix. The *permanental polynomial* is similar to the characteristic polynomial, which is defined as [20]

$$\pi(G,x)=\operatorname{per}(xI-A(G))=\sum_{i=0}^{n} b_i x^{n-i},$$

where $\operatorname{per}(\cdot)$ denotes the permanent of a matrix.

The characteristic polynomial and permanental polynomial are important evaluations of a graph. Much work focuses on the coefficients and roots, as well as the relationships between permanental polynomials and characteristic polynomials [1–3,20–22]. It was shown that the coefficients of the characteristic and permanental polynomials of graphs are related to the graph structure [3,6,16]. A *linear subgraph* (or *basic figure*) of a graph G is termed as a subgraph whose components are cycles or single edges. A linear subgraph on i vertices is denoted by U_i. It holds

$$a_i=\sum_{U_i\subset G}(-1)^{p(U_i)}2^{c(U_i)} \quad \text{for } 1\le i\le n \tag{1}$$

☆ This work is supported by NSFC (Grant No. 11501448), the Natural Science Basic Research Plan in Shaanxi Province of China (Grant No. 2015JQ1025) and the Fundamental Research Funds for the Central Universities (Grant No. 3102015ZY071).

E-mail address: liw@nwpu.edu.cn.

http://dx.doi.org/10.1016/j.dam.2016.03.021

and

$$b_i = (-1)^i \sum_{U_i \subset G} 2^{c(U_i)} \quad \text{for } 1 \le i \le n, \tag{2}$$

where the summations range over all linear subgraphs U_i of G, $p(U_i)$ is the number of components of U_i and $c(U_i)$ is the number of cycles of U_i. Particularly, b_2 is the number of edges of G and $-b_3$ is twice the number of triangles of G. For a bipartite graph G, b_n is equal to the square of the number of perfect matchings of G [15], i.e.

$$b_n = m^2(G). \tag{3}$$

Permanents and permanental polynomials are not only related to some interesting theoretical problems in graph theory and combinatorics [13,17], but also have been used as invariants for chemical structures [4,5,10]. Recently, the skew-permanental polynomials and skew-characteristic polynomials have attracted some attention [9,12,14]. They are defined in terms of orientation graphs.

An *orientation graph* G^σ of a graph G is an assigning of directions to the edges of G, called the orientation of G for short. The *skew-adjacency matrix* of G^σ is $S(G^\sigma) = (s_{ij})_{n\times n}$, where

$$s_{ij} = \begin{cases} 1 & \text{if there is an edge directed from } v_i \text{ to } v_j \text{ in } G^\sigma, \\ -1 & \text{if there is an edge directed from } v_j \text{ to } v_i \text{ in } G^\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

The *skew-characteristic polynomial* of an orientation graph G^σ is $\det(xI - S(G^\sigma))$, denoted as $\phi_s(G^\sigma, x) = \sum_{i=0}^{n} c_i x^{n-i}$. The *skew-permanental polynomial* of G^σ is $\mathrm{per}(xI - S(G^\sigma))$, written as $\pi_s(G^\sigma, x) = \sum_{i=0}^{n} d_i x^{n-i}$.

Some research revealed that the coefficients of the skew-characteristic polynomial and skew-permanental polynomial can be expressed by the structure of subgraphs [9,14]. As the coefficients of these polynomials contain rich combinatorial information, the study of connections on the coefficients of different polynomials provides new avenues to investigate graph parameters and graph structures.

In an orientation graph, the existence of -1 in the skew-adjacency matrix leads to $|d_n| \le m^2(G)$. When does the equality hold? Our first problem is finding an orientation of G satisfying $|d_n| = m^2(G)$ (it means $|d_n| = |b_n|$). Furthermore, we consider which orientation of G satisfies $|d_i| = |b_i|$ for each i. It has been proved that the permanental polynomials of some graphs can be expressed by the skew-characteristic polynomials of orientation graphs [21]. Following this line, we pay attention to the problem if there is an orientation G^σ of G such that the skew-permanental polynomial of G^σ is equal to the characteristic polynomial of G.

The organization of this paper is as follows. In Section 2, we give a sufficient and necessity condition for the orientation graphs with $|d_n| = m^2(G)$, and provide a characterization of such orientation graphs. In Section 3, we further characterize the orientation graph with $|d_i| = |b_i|$ for each i. To calculate the skew-permanental polynomials, in Section 4 we provide three equalities to express the skew-permanental polynomial of an orientation graph in terms of the ones of subgraphs. In Section 5, we characterize the graph whose skew-permanental polynomial can be computed by the corresponding characteristic polynomial.

2. The constant term of the skew-permanental polynomial

For a bipartite graph, we consider the relationship between the number of perfect matchings and the constant term of the skew-permanental polynomial of the corresponding orientation graph. We mainly characterize those orientation graphs with $|d_n| = m^2(G)$. For our proofs we need some definitions and notations.

An *even cycle* (resp. *odd cycle*) means a cycle of even length (resp. odd length). An even cycle in an orientation graph is *oddly oriented* (resp. *evenly oriented*) if there is an odd (resp. even) number of edges directed in either direction of the cycle. An *even linear subgraph* U_i^e is a subgraph on i vertices whose components are single edges or even cycles. In view of a linear subgraph of a bipartite graph is an even linear subgraph, we will not distinguish them.

For the permanental and characteristic polynomials of orientation graphs, the following results on coefficients c_i and d_i were proved in [9] and [14], respectively.

Theorem 2.1 *([9,14]). Let G^σ be an orientation of a graph G. Then*

$$c_i = \sum_{U_i^e \subset G^\sigma} (-1)^{c^+(U_i^e)} 2^{c(U_i^e)} \quad \textit{for } i \textit{ is even} \tag{4}$$

and

$$d_i = \sum_{U_i^e \subset G^\sigma} (-1)^{s(U_i^e)+c^-(U_i^e)} 2^{c(U_i^e)} \quad \textit{for } i \textit{ is even,} \tag{5}$$

where the sums range over all the even linear subgraphs U_i^e of G^σ. Moreover, $s(U_i^e)$ is the number of components of U_i^e which are single edges, $c(U_i^e)$ is the number of cycles of U_i^e and $c^-(U_i^e)$ (resp. $c^+(U_i^e)$) is the number of oddly (resp. evenly) oriented cycles of U_i^e. In particular, $c_i = d_i = 0$ for i is odd and $c_0 = d_0 = 1$.