Discrete Applied Mathematics 172 (2014) 12–27

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On disjoint matchings in cubic graphs: Maximum 2-edge-colorable and maximum 3-edge-colorable subgraphs

Davit Aslanyan [a], Vahan V. Mkrtchyan [a,b,*], Samvel S. Petrosyan [a], Gagik N. Vardanyan [a]

[a] Department of Informatics and Applied Mathematics, Yerevan State University, Yerevan, 0025, Armenia

[b] Institute for Informatics and Automation Problems, National Academy of Sciences of Republic of Armenia, 0014, Armenia

ARTICLE INFO

Article history:
Received 10 October 2012
Received in revised form 12 February 2014
Accepted 4 March 2014
Available online 24 March 2014

Keywords:
Edge-coloring
Cubic graph
Maximum 2-edge-colorable subgraph
Maximum 3-edge-colorable subgraph

ABSTRACT

We show that any 2-factor of a cubic graph can be extended to a maximum 3-edge-colorable subgraph. We also show that the sum of sizes of maximum 2- and 3-edge-colorable subgraphs of a cubic graph is at least twice of its number of vertices. Finally, for a cubic graph G, consider the pairs of edge-disjoint matchings whose union consists of as many edges as possible. Let H be the largest matching among such pairs. Let M be a maximum matching of G. We show that $9/8$ is a tight upper bound for $|M|/|H|$.

1. Introduction

We consider finite undirected graphs that do not contain loops. Graphs may contain multiple edges. For a graph G and a positive integer k define

$$B_k(G) = \{\{H_1, \ldots, H_k\} : H_1, \ldots, H_k \text{ are pairwise edge-disjoint matchings of } G\},$$

and let

$$\nu_k(G) = \max\{|H_1| + \cdots + |H_k| : \{H_1, \ldots, H_k\} \in B_k(G)\}.$$

A subgraph H of G is called maximum k-edge-colorable, if it is k-edge-colorable and contains exactly $\nu_k(G)$ edges. If G is edge-colored, then for a vertex v of G let $C(v)$ denote the set of colors of the edges of G that are incident to the vertex v.

For a graph G define:

$$\alpha_k(G) = \max\{|H_1|, \ldots, |H_k| : \{H_1, \ldots, H_k\} \in B_k(G) \text{ and } |H_1| + \cdots + |H_k| = \nu_k(G)\}.$$

If $\nu(G)$ denotes the cardinality of the largest matching of G, then it is clear that $\alpha_k(G) \leq \nu(G)$ for all G and k. Moreover, $\nu_k(G) = |E(G)|$ for all $k \geq \chi'(G)$, where $\chi'(G)$ is the chromatic index of G. Also note that $\nu_1(G)$ and $\alpha_1(G)$ are equal to $\nu(G)$.

Recall that a matching of G is maximum, if it contains $\nu(G)$ edges, and is maximal if it is not a subset of a larger matching. In contrast with the theory of 2-matchings, where every graph G admits a maximum 2-matching that includes a maximum matching [4], there are graphs that do not have a maximum 2-edge-colorable subgraph that includes a maximum matching.

* Corresponding author at: Department of Informatics and Applied Mathematics, Yerevan State University, Yerevan, 0025, Armenia. Tel.: +374 93419589.
E-mail addresses: aslanyan@ymail.com (D. Aslanyan), vahanmkrtchyan2002@ipia.sci.am, vahanmkrtchyan2002@ysu.am, vahanmkrtchyan2002@yahoo.com (V.V. Mkrtchyan), samvelpetrosyan2008@yahoo.com (S.S. Petrosyan), vgagik@gmail.com (G.N. Vardanyan).

http://dx.doi.org/10.1016/j.dam.2014.03.001
0166-218X/

The following is the best result that can be stated about the ratio $\nu(G)/\alpha_2(G)$ for any simple graph G (see [7]):

$$1 \leq \nu(G)/\alpha_2(G) \leq 5/4.$$

A very deep characterization of simple graphs G satisfying $\nu(G)/\alpha_2(G) = 5/4$ is given in [11].

Also note that by Mkrtchyan's result [5], reformulated as in [3], if G is a matching covered tree, then $\alpha_2(G) = \nu(G)$. Note that a graph is said to be matching covered (see [6]), if each of its edges belongs to a maximum matching (not necessarily a perfect matching as it is usually defined, see e.g. [4]).

In this paper, we show that any 1- and 2-factor of a cubic graph can be extended to a maximum 3-edge-colorable subgraph. We also show that $\nu_2(G) + \nu_3(G) \geq 2|V(G)|$ for any cubic graph G. Finally, we show that 9/8 is a tight upper bound for the ratio $\nu(G)/\alpha_2(G)$ in the class of cubic graphs G.

Terms and concepts that we do not define can be found in [4,12].

2. The main results

We begin with a theorem that describes the structure of the edges that do not belong to a maximum 3-edge-colorable subgraph of a cubic graph.

Theorem 1. *Let H be a maximum 3-edge-colorable subgraph of a cubic graph G. Then $E(G) \setminus E(H)$ is a matching.*

Proof. To complete the proof of the theorem, we need to verify the absence of adjacent edges in $G \setminus E(H)$.

Suppose that $\{u_0, u_1\}, \{u_1, u_2\} \in E(G) \setminus E(H)$. We need to consider two cases:

Case 1: $u_0 = u_2$, that is, $\{u_0, u_1\}$ is a multiple edge. Note that $|C(u_0)| \leq 1$, $|C(u_1)| \leq 1$, thus there is $\alpha \in \{1, 2, 3\}$ with $\alpha \notin C(u_0) \cup C(u_1)$. Now, if we color one of edges connecting u_0 and u_1 with color α, then we would get a proper 3-edge-coloring of the subgraph $H \cup \{\{u_0, u_1\}\}$, contradicting the maximality of H.

Case 2: $u_0 \neq u_2$. Note that $|C(u_0)| \leq 2$, $|C(u_1)| \leq 1$, $|C(u_2)| \leq 2$. It is easy to see that the maximality of H implies that

$$C(u_0) \cup C(u_1) = \{1, 2, 3\} \quad \text{and} \quad C(u_1) \cup C(u_2) = \{1, 2, 3\},$$

thus $|C(u_0)| = 2$, $|C(u_1)| = 1$, $|C(u_2)| = 2$ and $C(u_0) = C(u_2)$. Suppose that $C(u_0) = C(u_2) = \{\alpha, \beta\}$ and $C(u_1) = \{\gamma\}$. Consider the maximal $\alpha - \gamma$ alternating paths P_0, P_1, P_2, starting from vertices u_0, u_1, u_2, respectively. Note that there is $i \in \{0, 2\}$ such that $u_1 \notin V(P_i)$. Now, shift the colors on the path P_i to obtain a new coloring of the maximum 3-edge-colorable subgraph H, where the color α is absent in both of vertices u_i and u_1. Now, if we color the edge $\{u_1, u_i\}$ with color α, then we would get a proper 3-edge-coloring of the subgraph $H \cup \{\{u_1, u_i\}\}$, contradicting the maximality of H. The proof of Theorem 1 is completed. □

It is not always possible to extend a 1-factor (and maximum matchings as well [1]) to a maximum 2-edge-colorable subgraph of a cubic graph. Nevertheless, the following is true:

Theorem 2. *Any 1-factor of a cubic graph G can be extended to a maximum 3-edge-colorable subgraph of G.*

Proof. For a 1-factor F of G, choose a maximum 3-edge-colorable subgraph H of G with $|E(F) \cap E(H)|$ is maximum.

Let us show that $E(F) \subseteq E(H)$. On the opposite assumption, consider an edge $e = \{u, v\} \in E(F) \setminus E(H)$ and assume that H is properly colored with colors $\{1, 2, 3\}$. Due to Theorem 1, the edges adjacent to e belong to H. Note that the maximality of H implies that

$$|C(u) \cap C(v)| = 1 \quad \text{and} \quad C(u) \cup C(v) = \{1, 2, 3\}.$$

Choose $\alpha \in C(u) \setminus C(v)$. Consider the subgraph $H' = (H \setminus \{e'\}) \cup \{e\}$, where e' is the edge that is incident to u and is colored by α. Note that H' is a maximum 3-edge-colorable subgraph of G with

$$|E(F) \cap E(H)| < |E(F) \cap E(H')|$$

contradicting the choice of H. The proof of Theorem 2 is completed. □

Next, we prove a result which claims that the uncolored edges with respect to a maximum 3-edge-colorable subgraph of G always can be "left" in a given 1-factor, or, equivalently, any 2-factor of a cubic graph G can also be extended to a maximum 3-edge-colorable subgraph of G.

Theorem 3. *Let F be any 1-factor of a cubic graph G, and let $\bar{F}$ be the complementary 2-factor of F. Then there is a maximum 3-edge-colorable subgraph H of G, such that:*

(a) $E(H) \cup E(F) = E(G)$;
(b) $E(\bar{F}) \subseteq E(H)$.

Proof. Note that (b) follows from (a), thus we will only prove (a).

For a given 1-factor F of a cubic graph G, consider a maximum 3-edge-colorable subgraph H of G such that $|E(F) \cap E(H)|$ is minimum.