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The complexity of pebbling reachability and solvability in planar and outerplanar graphs☆

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ABSTRACT

Given a simple, connected graph, a *pebbling configuration* is a function from its vertex set to the nonnegative integers. A *pebbling move* between adjacent vertices removes two pebbles from one vertex and adds one pebble to the other. A vertex r is said to be *reachable* from a configuration if there exists a sequence of pebbling moves that places at least one pebble on r. A configuration is *solvable* if every vertex is reachable. We prove that determining reachability of a vertex and solvability of a configuration are NP-complete on planar graphs. We also prove that both reachability and solvability can be determined in $O(n^6)$ time on planar graphs with diameter two. Finally, for outerplanar graphs, we present a linear algorithm for determining reachability and a quadratic algorithm for determining solvability. To prove this result, we provide linear algorithms to determine all possible maximal configurations of pebbles that can be placed on the endpoints of a path and on two adjacent vertices in a cycle.

1. Introduction

A graph $G = (V, E)$ is a set V of *vertices* and a set E of pairs of vertices called *edges*. The maximum number of edges in the shortest path between any two vertices is called the *diameter* of G. G is *planar* if it can be embedded in a plane, i.e. if it can be drawn such that no two edges intersect. G is *outerplanar* if it can be embedded in a plane so that every edge is incident to the unbounded face. The *dual* of an embedding of a planar graph is the graph obtained by adding a vertex for each face and connecting vertices whose corresponding faces share an edge. The *weak dual* is the induced subgraph of the dual obtained by removing the vertex corresponding to the unbounded face. It is well known that the weak dual of an outerplanar graph is a forest. If a vertex v is adjacent to at least one vertex in a set $A \subseteq V$, then we will say that v is *adjacent to* A. A *dominating set* $S \subseteq V$ has the property that every vertex not in S is adjacent to S. The *domination number* of G is the cardinality of the smallest dominating set in G.

A *pebbling configuration* (or just *configuration*) is a function $C : V(G) \to \mathbb{N}$ (the nonnegative integers) where $C(v)$ represents the number of "pebbles" placed on vertex v. The size of C, denoted $|C|$, is the sum of the pebbles on the vertices of G. For $k \in \mathbb{N}$ we define $V_k = \{v \in V \mid C(v) \geq k\}$. That is, V_k is the set of vertices that have at least k pebbles. A *pebbling move* from a vertex u to an adjacent vertex v, denoted (u, v), removes two pebbles from u and adds one pebble to v, assuming u

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has at least two pebbles. A vertex is *reachable* if it has at least one pebble assigned to it in the initial configuration C or it can receive a pebble through a sequence of pebbling moves. If every vertex in G is reachable, then C is said to be *solvable*. The *pebbling number* of G is the minimum integer k such that every pebble distribution of size k on G is solvable.

If ϕ is a sequence of pebbling moves, we define C_ϕ to be the configuration after the moves of ϕ have been performed on G with starting configuration C. A sequence of pebbling moves that accomplishes some goal (e.g. places one pebble on a specified vertex) is *minimal* if removing any moves renders the goal unattainable. A sequence is *minimum* if no other sequence accomplishes the same goal with less moves. If a pebbling sequence accomplishes some goal, some minimal sequence does as well. Thus, we assume throughout that all pebbling sequences are minimal.

Graph pebbling can be extended to weighted graphs by defining a weight function w that assigns a nonnegative integer to each edge [6,10,17]. A pebbling move from a vertex u to another vertex v removes $w(u, v)$ pebbles from u and adds one pebble to v, assuming $(u, v) \in E$ and u has at least $w(u, v)$ pebbles. Traditional pebbling on unweighted graphs can be thought of as a special case of pebbling on weighted graphs where the weight of each edge is 2. We further extend graph pebbling to directed weighted graphs using this definition of weight, with the only difference being that the direction of the edges matters. An alternative method of extending pebbling to weighted graphs is given in [12].

It has previously been shown that Reachable and Solvable are NP-complete [11,14,20], and that determining whether or not the pebbling number of a graph is at most k (Pebbling-Number) is Π_2^p-complete [14]. Techniques have been developed to compute the pebbling number for small graphs [17] and bounds on the pebbling number [9]. In addition, the pebbling number of graphs in a family known as split graphs can be computed in polynomial time [1].

For diameter two graphs, more is known. The pebbling number of a diameter two graph is either n or $n + 1$, and [4,3] imply that determining which it is can be accomplished in polynomial time. An explicit algorithm is given in [2], with a slight improvement given in [8]. Although Pebbling-Number is easier for diameter two graphs, Reachable remains NP-complete for diameter two graphs (D2-Reachable) [5]. However, D2-Reachable is decidable in polynomial time for diameter two graphs with pebbling number $n + 1$ or those with pebbling number n that have small connectivity [2].

Studying the complexity of Reachable restricted to planar and outerplanar graphs was suggested in [14]. We show that in the planar case, Reachable (P-Reachable) and Solvable (P-Solvable) remain NP-complete. However, Reachable and Solvable are both decidable in $O(|V|^6)$ time when restricted to planar graphs having diameter two (PD2-Reachable and PD2-Solvable). Finally, we present a linear algorithm for Reachable on outerplanar graphs (OP-Reachable), which implies that Solvable can be decided in $O(n^2)$ time on outerplanar graphs (OP-Solvable).

2. Planar graphs

First we will define Clause Cycle Planar 3SAT (CCP3SAT), a restricted form of 3SAT, and show that it is NP-complete. Then we will reduce from CCP3SAT to P-Reachable by constructing a planar graph using gadgets for each variable and clause so that under a certain pebbling configuration, a designated vertex r will be reachable if and only if the 3-CNF formula is satisfiable. We will construct variable gadgets with positive and negative edges corresponding to true and false assignments of the variables. To ensure our gadget enforces a valid truth assignment, each positive (resp. negative) edge can be pebbled along if and only if no pebbling moves have been made along any negative (resp. positive) edges. Then we will connect clause gadgets in a path so that each clause can only be pebbled from if it is pebbled to by at least one variable gadget and by the previous clause gadget. We will be able to pebble from the last clause to a specified target vertex if and only if each clause has a literal that evaluates to true.

2.1. Clause cycle planar 3SAT

Let F be a 3-CNF formula with n variables $\{v_1, \ldots, v_n\}$ and m clauses $\{c_1, \ldots, c_m\}$. Define $G(F) = (V, E)$, where $V = \{v_i \mid 1 \le i \le n\} \cup \{c_j \mid 1 \le j \le m\}$ and $E = \{(v_i, c_j) \mid v_i \in c_j \text{ or } \bar{v}_i \in c_j\} \cup \{(v_i, v_j) \mid j = (i + 1) \bmod n\}$. 3SAT restricted to instances of 3-CNF formula F where $G(F)$ is planar is called Planar 3SAT (P3SAT). P 3SAT was shown to be NP-Complete by Lichtenstein [13] by constructing an equivalent planar 3-CNF formula for a given general 3-CNF formula. The construction arranges clauses and variables in a grid (Fig. 1(a)) with the intersecting edges replaced with a crossover box (Fig. 2(a)). Then a cycle is drawn through the variables in the grid (Fig. 1(a)) that goes through each crossover box (Fig. 2(b)).

Given a 3-CNF formula F, define the graph $H(F) = (V, E)$, where $V = \{v_i \mid 1 \le i \le n\} \cup \{c_j \mid 1 \le j \le m\}$ and $E = \{(v_i, c_j) \mid v_i \in c_j \text{ or } \bar{v}_i \in c_j\} \cup \{(c_i, c_j) \mid j = (i + 1) \bmod m\}$. Define Clause Cycle Planar 3SAT (CCP3SAT) to be a subset of 3SAT where the graph $H(F)$ is planar. This is analogous to P3SAT except the clauses instead of the variables are connected by a cycle.

Theorem 1. *CCP3SAT is NP-complete.*

Proof. CCP3SAT is trivially in NP. We will show that CCP3SAT is NP-hard by showing that the 3-CNF formula reduced to in [13] is not only in P3SAT but also satisfies the requirements to be in CCP3SAT. Given a 3-CNF formula F, construct an equivalent 3-CNF formula F' using the reduction in [13]. Given the graph $G(F')$ from the reduction, remove the edges between variables in both the overall diagram (Fig. 1(a)) and the crossover boxes (Fig. 2(b)), and create a cycle between the clauses so that the graph remains planar as follows. First, draw a cycle going through all of the crossover boxes in the same order and by the same quadrants as [13] but connecting clauses instead of variables (Fig. 1(b)). Then create a path through the clauses