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Note

Graphs inducing totally balanced and submodular Chinese postman games

M. Josune Albizuri [a,*,1], Herbert Hamers [b]

[a] Department of Applied Economics IV, Faculty of Business and Economics, University of the Basque Country, Avda Lehendakari Agirre, 83, 48015 Bilbao, Spain

[b] CentER and Department of Econometrics and OR, TiSEM and TIAS School for Business and Society, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands

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ABSTRACT

A Chinese postman (CP) game is induced by a weighted undirected, connected graph in which the edges are identified as players and a vertex is chosen as post-office location.

Granot and Granot (2012) characterized graphs that give rise to CP games that are balanced. This note completes this line of research by characterizing graphs that give rise to CP games that are submodular (totally balanced, respectively).

1. Introduction

Chinese postman (CP) games, introduced by Hamers et al. [8], are defined on a weighted undirected connected graph in which a vertex is fixed, referred to as post-office, and the players reside on the edges. More precisely, the choice of the location of the post-office and the non-negative weighted (or cost) function determines a specific CP game on this graph, since the value of a coalition in a Chinese postman (CP) game is obtained by a cheapest tour that starts and ends at the post-office and visits all members of this coalition. Hence, the value of a coalition reflects the cheapest cost a coalition can be visited. Observe that the cost of the cheapest tour that visits all edges in a graph at least once is equal to the value of the grand coalition, i.e. the set that consists of all players, of the CP game that is induced by that graph. Hence, the value of the grand coalition is the result of solving the related Chinese postman (CPP) problem (cf. [11,2]).

For a cooperative game (N, c), where N is the set of players and $c : 2^N \rightarrow \mathbb{R}$ is the characteristic function, the core of the cooperative game (cf. [3]) consists of all vectors that distribute the cost of the grand coalition among the players in such a way that no subset of the grand coalition has an incentive to deviate from the grand coalition, i.e.

$$Core(N, c) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i \in S} x_i \leq c(S) \text{ for all } S \subset N, \quad \sum_{i \in N} x_i = c(N) \right\}.$$

Hamers [7] showed that CP games may not be balanced, i.e. the core is empty. However, he showed that a CP game is balanced if the corresponding graph is weakly Eulerian. Further, he showed that a CP game is submodular if the corresponding graph is weakly cyclic, i.e. every edge in this graph is contained in at most one circuit. A game (N, c) is called submodular if

* Corresponding author.
E-mail addresses: mj.albizuri@ehu.es (M.J. Albizuri), H.J.M.Hamers@tilburguniversity.edu (H. Hamers).

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its characteristic function is submodular, i.e. for all $S, T \subset N$ holds

$$c(S \cup T) + c(S \cap T) \leq c(S) + c(T).$$

A game (N, c) is called totally balanced if each subgame (T, c_T) is balanced, where $c_T(S) = c(S)$ for all $S \subseteq T$. Observe that submodular games are totally balanced (cf. [14]), and totally balanced games are balanced.

Granot et al. [6] called a graph global CP balanced (totally balanced, submodular, respectively), if for all locations of the post-office and for all non-negative weight functions the corresponding CP game is balanced (totally balanced, submodular, respectively). They showed that a graph is CP balanced if and only if the graph is weakly Eulerian, and a graph is CP submodular if and only if it is CP totally balanced if and only if it is weakly cyclic.

Granot and Hamers [5] called a graph locally CP balanced (totally balanced, submodular, respectively), if there exists at least one location of the post-office and for all non-negative weight functions the corresponding CP game is balanced (totally balanced, submodular, respectively). They showed that the locally CP balanced (totally balanced, submodular, respectively) graphs coincide with the globally balanced (totally balanced, submodular,respectively) graphs.

A graph is called super locally CP balanced (totally, submodular, respectively) if for at least one location of the post-office and for at least one positive weight function the corresponding CP game is balanced (totally balanced, submodular), which is a direct relaxation of the definitions introduced in [6,5]. Observe that the notion of super locally CP balanced graphs and the class of CP core-nonempty graphs, as defined in [4], define different kinds of sets. In fact, super locally CP balanced graphs described a class of graphs, whereas CP core-nonempty graphs define a class of graphs in which one vertex is fixed (as post-office). Nevertheless, each graph that can become a CP core-nonempty graph after identifying a vertex as post office is a super locally CP balanced graph. So, in this respect we can also consider this paper as a completion of the line of research as presented in [4].

Totally balancedness is an interesting property since for each subgame a core element can be provided and from the perspective of Population Monotonic Allocation Schemes (PMAS), introduced by Sprumont [16], since it is a necessary condition for the existence of a PMAS. The significance of submodularity is even more important since for these games some solution concepts have nice properties. For instance, the Shapley value is the barycenter of the core [15], the Aumann–Davis–Maschler bargaining set coincides with the core and the nucleolus coincides with the kernel [12] and the compromise value [17] can be calculated in polynomial time.

CP games are contained in the class of OR games arising from network problems. There exists a line of research in which game theoretical properties of the OR game are characterized by properties of the underlying network (graph). For example, [10] showed that graphs which are obtained as a 1-sum of K_4 and outerplanar graphs characterize submodular Steiner-traveling salesman games. Okamoto [13] showed that minimum vertex cover games are submodular if and only if the underlying graph is (K_3, P_3)-free, i.e., no induced subgraph is isomorphic to K_3 or P_3 and minimum coloring games are submodular if and only if the underlying graph is complete multipartite. Deng et al. [1] showed that minimum coloring games are totally balanced if and only if the underlying graph is perfect. Hamers et al. [9] showed that minimum coloring games have a Population Monotonic Allocation scheme if and only if the graph is $(P_4, 2K_2)$-free.

This note introduces in Section 2 besides some notions from graph theory, the Chinese postman game. The characterizations of super locally CP submodular and super locally CP totally balanced graphs are presented in Section 3, respectively.

2. Chinese postman games

Let $G = (V(G), E(G))$ be an undirected graph where $V(G)$ and $E(G)$ denote the set of vertices and edges of G respectively. Let $v_0 \in V(G)$ denote the post-office in G. A *walk* in $G = (V(G), E(G))$ is a finite sequence of vertices and edges of the form $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ with $k \geq 0$, $v_1, v_2, \ldots, v_{k+1} \in V(G)$ and $e_1, e_2, \ldots, e_k \in E(G)$ such that $e_j = (v_j, v_{j+1})$ for all $j \in \{1, \ldots, k\}$. If $v_1 = v_{k+1}$ then the walk is referred to as *a closed walk* and if $v_1 = v_{k+1} = v_0$ as a *tour*. If all edges of a walk are different then the walk is a *path*. The graph $G = (V(G), E(G))$ is *connected* if for any two vertices in G there is a path in G between the two vertices. A closed walk containing at least one edge is called a *circuit*. A graph is said to be *weakly cyclic* if it is connected and every edge therein is contained in at most one circuit.

Let $l : E(G) \to \mathbb{R}_{++}$ be a positive *edge-cost function*. We assume that each edge belongs to a different player. Therefore, the set of players $N(G)$ can be identified with the edge set $E(G)$, i.e. $N(G) = E(G)$.

Let $T = v_0, e_1, \ldots, e_k, v_0$ be a tour in G. Then T is *feasible* for a coalition $S \subseteq E(G)$ if every edge of S is visited by T, i.e. $S \subseteq \{e_1, \ldots, e_k\}$. The total cost of T is $k(T) = \sum_{j=1}^{k} l(e_j)$. Given an edge set $E_1 \subset E(G)$ we denote by $k(E_1)$ the costs of the edges in E_1, i.e. $k(E_1) = \sum_{e \in E_1} l(e)$, and the costs of a path P equals $k(P) = \sum_{e \in E(P)} l(e)$ where $E(P)$ are the edges of path P.

Definition 2.1. The Chinese Postman (CP) game, $(N(G), c)$, induced by a connected graph $G = (V(G), E(G))$, in which $v_0 \in V(G)$ is the post-office and l the positive edge-cost function is defined by

$$c_{l_G}(S) = \min\{k(T) : T \text{ is a feasible tour for } S\}$$

for every $S \subseteq N(G)$.

From now on, for short, we will say that (G, v_0, l) induces the CP game $(N(G), c_{l_G})$.