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Graphs with two trivial critical ideals

Carlos A. Alfaro, Carlos E. Valencia *

Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del IPN, Apartado Postal 14–740, 07000 Mexico City, D.F., Mexico

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ABSTRACT

The critical ideals of a graph are the determinantal ideals of the generalized Laplacian matrix associated to a graph. A basic property of the critical ideals of graphs asserts that the graphs with at most k trivial critical ideals, $\Gamma_{\leq k}$, are closed under induced subgraphs. In this article we find the set of minimal forbidden subgraphs for $\Gamma_{\leq 2}$, and we use this forbidden subgraphs to get a classification of the graphs in $\Gamma_{\leq 2}$. As a consequence we give a classification of the simple graphs whose critical group has two invariant factors equal to one. At the end of this article we give two infinite families of forbidden subgraphs.

1. Introduction

Given a connected graph $G = (V(G), E(G))$ and a set of indeterminates $X_G = \{x_u \mid u \in V(G)\}$, the *generalized Laplacian matrix* $L(G, X_G)$ of G is the matrix with rows and columns indexed by the vertices of G given by

$$L(G, X_G)_{uv} = \begin{cases} x_u & \text{if } u = v, \\ -m_{uv} & \text{otherwise,} \end{cases}$$

where m_{uv} is the *multiplicity* of the edge uv, that is, the number of the edges between vertices u and v of G. For all $1 \leq i \leq n$, the *i-critical ideal* of G is the determinantal ideal given by

$$I_i(G, X_G) = \langle \{\det(m) \mid m \text{ is a square submatrix of } L(G, X_G) \text{ of size } i\} \rangle \subseteq \mathbb{Z}[X_G].$$

We say that a critical ideal is trivial when it is equal to $\langle 1 \rangle$.

Critical ideals are a generalization of the characteristic polynomials of the adjacency matrix and the Laplacian matrix associated to a graph. Also, critical ideals generalize the critical group of a graph as shown below: if $d_G(u)$ is the degree of a vertex u of G, then the *Laplacian matrix* of G, denoted by $L(G)$, is the evaluation of $L(G, X_G)$ on $x_u = d_G(u)$. Given a vertex s of G, the *reduced Laplacian matrix* of G, denoted by $L(G, s)$, is the matrix obtained from $L(G)$ by removing the row and column s. The *critical group* of a connected graph G, denoted by $K(G)$, is the cokernel of $L(G, s)$. That is,

$$K(G) = \mathbb{Z}^{\widetilde{V}} / \operatorname{Im} L(G, s),$$

where $\widetilde{V} = V(G) \setminus s$. Therefore the critical group of a graph can be obtained from their critical ideals as shows [4, Theorems 3.6 and 3.7]. Note that the critical group $K(G)$ of a graph G does not depend of the vertex s, see [3, Proposition 1.1]. The critical group have been studied intensively on several contexts over the last 30 years. However, a well understanding of the combinatorial and algebraic nature of the critical group still remains.

* Corresponding author. Tel.: +52 55 57473800x6433; fax: +52 55 57473876.
E-mail addresses: alfaromontufar@gmail.com (C.A. Alfaro), cvalencia@math.cinvestav.edu.mx, cvalencia75@gmail.com (C.E. Valencia).

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Let assume that G is a connected graph with n vertices. A classical result (see [6, Section 3.7]) asserts that the reduced Laplacian matrix is equivalent to a integer diagonal matrix with entries $d_1, d_2, \dots, d_{n-1}$ where $d_i > 0$ and $d_i \mid d_j$ if $i \le j$. The integers $d_1, \dots, d_{n-1}$ are unique and are called *invariant factors*. With this in mind, the critical group is described in terms of the invariant factors as follows [8, Theorem 1.4]:

$$K(G) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_{n-1}}.$$

Given an integer number k, let $f_k(G)$ be the number of invariant factors of the Laplacian matrix of G equal to k. Let $G_i = \{G \mid G \text{ is a simple connected graph with } f_1(G) = i\}$. The study and characterization of G_i is of great interest. In particular, some results and conjectures on the graphs with cyclic critical group can be found in [10, Section 4] and [13, Conjectures 4.3 and 4.4]. On the other hand, Dino Lorenzini, notice in [9] that G_1 consists only of the complete graphs. More recently, Merino in [11] posed interest on the characterization of G_2 and G_3. In this sense, few attempts have been done. For instance, in [12] it was characterized the graphs in G_2 whose third invariant factor is equal to n, $n-1$, $n-2$, or $n-3$. In [2] the characterizations of the graphs in G_2 with a cut vertex, and the graphs in G_2 with number of independent cycles equal to $n-2$ are given.

Let $\Gamma_{\le i}$ denote the family of graphs with at most i trivial critical ideals. It is not difficult to see that $G_i \subseteq \Gamma_{\le i}$ for all $i \ge 0$. At first glance, critical ideals behave better than critical groups. For instance, by [4, Proposition 3.3] we have that $\Gamma_{\le i}$ is closed under induced subgraphs compared to G_i. This property will play a crucial role in order to get a characterization of $\Gamma_{\le 2}$ on this paper. Also, if Γ_i is the family of graphs with exactly i trivial critical ideals, then we will shown on this paper that Γ_2 has a more simple description than G_2.

There are three main goals of this paper: to get a characterization of the graphs with at most two trivial critical ideals, to get a characterization of the graphs with two invariant factors equal to one, and to give two infinite families of forbidden subgraphs for $\Gamma_{\le i}$.

This article is divided as follows: We begin by recalling some basic concepts on graph theory in Section 2 and establishing some of basic properties of critical ideals in Section 3. In Section 4 we will characterize the graphs with at most two trivial critical ideals by finding their minimal set of forbidden graphs. As consequence, we will get the characterization of the graphs with two invariant factors equal to one. Finally, in Section 5 we give two infinite families of forbidden graphs for $\Gamma_{\le i}$.

2. Basic definitions

In this section, we give some basic definitions and notation of graph theory used in later sections. It should be pointed that we will consider the natural number as the non-negative integers.

Given a graph $G = (V, E)$ and a subset U of V, the subgraph of G *induced* by U will be denoted by $G[U]$. If u is a vertex of G, let $N_G(u)$ be the set of *neighbors* of u in G. Here a *clique* of a graph G is a maximum complete subgraph, and its order is the *clique number* of G, denoted by $\omega(G)$. The *path* with n vertices is denoted by P_n, a *matching* with k edges by M_k, the *complete graph* with n vertices by K_n and the *trivial graph* of n vertices by T_n. The *cone* of a graph G is the graph obtained from G by adding a new vertex, called *appex*, which is adjacent to each vertex of G. The cone of a graph G is denoted by $c(G)$. Thus, the *star* S_k of $k+1$ vertices is equal to $c(T_k)$. Given two graphs G and H, their *union* is denoted by $G \cup H$, and their *disjoint union* by $G+H$. The *join* of G and H, denoted by $G \vee H$, is the graph obtained from $G+H$ when we add all the edges between vertices of G and H. For $m, n, o \ge 1$, let $K_{m,n,o}$ be the *complete tripartite graph*. The reader can consult [5] for any unexplained concept of graph theory.

Let $M \in M_n(\mathbb{Z})$ be a $n \times n$ matrix with entries on $\mathbb{Z}$, $I = \{i_1, \dots, i_r\} \subseteq \{1, \dots, n\}$, and $J = \{j_1, \dots, j_s\} \subseteq \{1, \dots, n\}$. The submatrix of M formed by rows $i_1, \dots, i_r$ and columns $j_1, \dots, j_s$ is denoted by $M[I; J]$. If $|I| = |J| = r$, then $M[I; J]$ is called a *r-square submatrix* or a *square submatrix* of size r of M. A *r-minor* is the determinant of a r-square submatrix. The set of i-minors of a matrix M will be denoted by $\text{minors}_i(M)$. Finally, the identity matrix of size n is denoted by I_n and the all ones $m \times n$ matrix is denoted by $J_{m,n}$. We say that $M, N \in M_n(\mathbb{Z})$ are *equivalent*, denoted by $N \sim M$, if there exist $P, Q \in GL_n(\mathbb{Z})$ such that $N = PMQ$. Note that if $N \sim M$, then $K(M) = \mathbb{Z}^n / M^t \mathbb{Z}^n \cong \mathbb{Z}^n / N^t \mathbb{Z}^n = K(N)$.

3. Graphs with few trivial critical ideals

In this section, we will introduce the critical ideals of a graph and the set of graphs with k or less trivial critical ideals, denoted by $\Gamma_{\le k}$. After that, we define the set of minimal forbidden graphs of $\Gamma_{\le k}$. We finish this section with the classification of G_1, that we already know that they are the complete graphs.

Let G be a graph and $X_G = \{x_v \mid v \in V(G)\}$ be the set of indeterminates indexed by the vertices of G. For all $1 \le i \le n$, the *i-critical ideal* $I_i(G, X_G)$ is defined as the ideal of $\mathbb{Z}[X_G]$ given by

$$I_i(G, X_G) = \langle \{\det(m) \mid m \text{ is a square matrix of } L(G, X_G) \text{ of size } i\} \rangle.$$

By convention $I_i(G, X_G) = \langle 1 \rangle$ if $i < 1$, and $I_i(G, X_G) = \langle 0 \rangle$ if $i > n$. The *algebraic co-rank* of G, denoted by $\gamma(G)$, is the number of critical ideals of G equal to $\langle 1 \rangle$.

Definition 3.1. For all $k \in \mathbb{N}$, let $\Gamma_{\le k} = \{G | G \text{ is a simple connected graph with } \gamma(G) \le k\}$ and $\Gamma_{\ge k} = \{G \mid G \text{ is a simple connected graph with } \gamma(G) \ge k\}$.

Note that, $\Gamma_{\le k}$ and $\Gamma_{\ge k+1}$ are disjoint sets and that a characterization of one of them leads to a characterization of the other one. Now, let us recall some basic properties about critical ideals, see [4] for details. It is known that if $i \le j$, then $I_j(G, X_G) \subseteq$