Discrete Applied Mathematics 167 (2014) 72–79

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Upper bounds on the average eccentricity

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ARTICLE INFO

Article history:
Received 28 July 2012
Received in revised form 16 November 2013
Accepted 25 November 2013
Available online 11 December 2013

Keywords:
Average eccentricity
Independence number
Chromatic number
Domination number
Connected domination number

ABSTRACT

Sharp upper bounds on the average eccentricity of a connected graph of given order in terms of its independence number, chromatic number, domination number or connected domination number are given. Our results settle two conjectures of the computer program AutoGraphiX (Aouchiche et al., 2005).

1. Introduction

Let $G = (V, E)$ be a connected graph of order n. The *distance* $d_G(u, v)$ between vertices u and v in G is the length of a shortest u–v path in G. The *eccentricity* $\mathrm{ec}_G(v)$ of a vertex v of G is the distance from v to a vertex furthest away from v. The *average eccentricity* $\mathrm{avec}(G)$ of G is defined as $\mathrm{avec}(G) = \frac{1}{n}\sum_{x\in V}\mathrm{ec}_G(x)$.

In this paper we give sharp upper bounds on the average eccentricity for graphs of given order and independence number, domination number, connected domination number or chromatic number. We prove, as corollaries to our results, two conjectures produced by the computer program AutoGraphiX.

The average eccentricity was introduced by Buckley and Harary [5] as the eccentric mean, and further studied in [10], where bounds in terms of order, radius, diameter and minimum degree were given, and for more recent results see [13]. The average eccentricity has also been the subject of a number of conjectures produced by the computer program AutoGraphiX. For information on the program AutoGraphiX see [2, 4, 6]; and for a list of its conjectures see [1, 11].

The average eccentricity is related to the average distance, defined as the average of the distances between all pairs of vertices. Many extremal graphs that maximise the average distance for a given parameter set also maximise the average eccentricity, and the results in this paper are a case in point. The average distance is also related to some new graph invariants studied in chemical graph theory, specifically to the eccentric connectivity index (see for example [10, 13] and references therein) and the eccentric distance sum (see for example [18]).

All graphs in this paper are finite and connected. Our notation follows [5]. In particular we denote the order, independence number, chromatic number, domination number and connected domination number of a graph G by n, α, χ, γ and γ_c. We denote the sum $\sum_{x\in V}\mathrm{ec}_G(x)$ by $\zeta(G)$, so $\mathrm{avec}(G) = \frac{\zeta(G)}{n}$. In most statements and proofs we use $\zeta(G)$, rather than $\mathrm{avec}(G)$,

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http://dx.doi.org/10.1016/j.dam.2013.11.023

in order to avoid unpleasant fractions. If x and y are vertices of a connected graph, then $P(x, y)$ denotes a shortest path between them.

2. Preliminary results

We begin with a few preparatory results which will be needed for the proofs of our main results.

The first proposition derives from the fact that the path maximises average eccentricity amongst all connected graphs of a given order, which was proved in [10]. The second proposition, a sharp upper bound on the average eccentricity of a tree of given order and diameter, is a refinement of the first. The third result is a classical result on eccentric sequences by Lesniak [14].

Proposition 2.1 ([10]). *Let G be a connected graph of order n. Then*

$$\zeta(G) \le \left\lfloor \frac{3}{4}n^2 - \frac{1}{2}n \right\rfloor,$$

with equality if and only if G is a path. $\square$

Lemma 2.1. *Let T be a tree and let u, v be two vertices at distance $\mathrm{diam}(T)$. Then*

$$\mathrm{ec}(x) = \max\{d(x, u), d(x, v)\}$$

for all $x \in V(T)$.

Proof. Suppose not. Then there exist vertices x and y with $d(x, y) > \max\{d(x, u), d(x, v)\}$.

Case 1: $P(u, v)$ and $P(x, y)$ have at most one vertex in common.

Let w and z be vertices on $P(u, v)$ and $P(x, y)$, respectively, at minimum distance, so $w = z$ if and only if the two paths have a vertex in common. By our assumption we have $d(z, u), d(z, v) < d(z, y)$, and so $d(w, y) > d(w, u), d(w, v)$. Since a shortest (u, y)-path contains w, we obtain the contradiction $d(u, y) = d(u, w) + d(w, y) > d(u, w) + d(w, v) = \mathrm{diam}(T)$.

Case 2: $P(u, v)$ and $P(x, y)$ share more than one vertex.

Then the two paths have a path segment $P(w, z)$ in common. We may assume that w is closer to u on $P(u, v)$ and z is closer to v. As above, $d(x, y) > d(x, v)$ implies $d(z, y) > d(z, v)$, and so we obtain the contradiction $d(u, y) = d(u, z) + d(z, y) > d(u, z) + d(z, v) = d(u, v) = \mathrm{diam}(T)$. $\square$

Proposition 2.2. *Let T be a tree of order n and diameter d. Then*

$$\zeta(T) \le \left\lfloor dn - \frac{1}{4}d^2 + \frac{1}{4} \right\rfloor.$$

Proof. Let $P = v_0, v_1, \ldots, v_d$ be a diametral path. By Lemma 2.1 we have $\mathrm{ec}(v_i) = d - i$ for $i \le \frac{1}{2}d$ and $\mathrm{ec}(v_i) = i$ for $i > \frac{1}{2}d$. Summation yields $\sum_{x \in V(P)} \mathrm{ec}(x) \le \lfloor \frac{3}{4}(d+1)^2 - \frac{1}{2}(d+1) \rfloor$. Hence,

$$\begin{aligned}
\zeta(T) &= \sum_{x \in V(P)} \mathrm{ec}(x) + \sum_{x \in V(T)-V(P)} \mathrm{ec}(x) \\
&\le \left\lfloor \frac{3}{4}(d+1)^2 - \frac{1}{2}(d+1) \right\rfloor + (n-d-1)d \\
&= \left\lfloor dn - \frac{1}{4}d^2 + \frac{1}{4} \right\rfloor,
\end{aligned}$$

as desired. $\square$

Theorem 2.1 ([14]). *Let G be a connected graph of order n. Then for every integer k with $\mathrm{rad}(G) < k < \mathrm{diam}(G)$ there exist at least two vertices in G of eccentricity k.* $\square$

Lemma 2.2. *Let G be a connected graph of order n, and let T be a connected dominating subgraph of G with vertex set S. Then*

$$\zeta(G) \le \zeta(T) + |S| + \Big(|V(G)| - |S|\Big)\Big(\mathrm{diam}(T) + 2\Big).$$

Proof. Since every vertex of G not contained in S is adjacent to a vertex in S, we have $\mathrm{ec}_G(x) \le \mathrm{ec}_T(x) + 1$ for all $x \in S$. Moreover, since every two vertices of G are connected by a path whose internal vertices are in S, we have $\mathrm{ec}_G(x) \le$