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Bondage number of grid graphs

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ABSTRACT

The bondage number $b(G)$ of a nonempty graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with domination number greater than the domination number of G. Here we study the bondage number of some grid-like graphs. In this sense, we obtain some bounds or exact values of the bondage number of some strong product and direct product of two paths.

1. Introduction

Let $G = (V, E)$ be a connected undirected graph with vertex set V and edge set E. Given two vertices $u, v \in V$, the notation $u \sim v$ means that u and v are adjacent. The *neighborhood* of a vertex $v \in V$ in G is the set $N_G(v) = \{u \in V : u \sim v\}$. For a set $X \subseteq V$, *the open neighborhood* $N_G(X)$ is defined to be $\bigcup_{v\in X} N_G(v)$ and *the closed neighborhood* $N_G[X] = N_G(X) \cup X$.

The *degree* $d_G(v)$ of a vertex v is the number of edges incident to v, $d_G(v) = |N_G(v)|$. The minimum and maximum degrees among all vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *distance* $d_G(u, v) = d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $(u - v)$ path in G.

A set $D \subseteq V$ is a *dominating set* of G if $N_G[D] = V$. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in G. Any dominating set of cardinality $\gamma(G)$ is called a γ-set. For unexplained terms and symbols see [7].

The *bondage number* $b(G)$ of a nonempty graph G with $E \neq \emptyset$ is the minimum cardinality among all sets of edges $E' \subseteq E$ for which $\gamma(G - E') > \gamma(G)$. The domination number of every spanning subgraph of a nonempty graph G is at least as great as $\gamma(G)$, hence the bondage number of a nonempty graph is well defined. Bondage number was introduced by Fink et al. [3] in 1990. However, the early research on the bondage number can be found in Bauer et al. [1]. In [1,3] it was shown that every tree has bondage number equal to 1 or 2. Hartnell and Rall [5] proved that for the cartesian product $G_n = K_n \square K_n$, $n > 1$, we have $b(G_n) = \frac{3}{2}\Delta$. Teschner [14,13,12] also studied the bondage number; for instance, in [13] he showed that $b(G) \leq \frac{3}{2}\Delta(G)$ holds for any graph G satisfying $\gamma(G) \leq 3$. Moreover, the bondage number of planar graphs was described in [2,4,10]. Carlson and Develin [2] showed that the corona $G = H \circ K_1$ satisfies $b(G) = \delta(H) + 1$. In [9] Kang et al. proved for discrete torus $C_n \square C_4$ that $b(C_n \square C_4) = 4$ for any $n \geq 4$. Also, some relationships between the connectivity and the bondage number of graphs were studied in [11]. In [8], the exact values of bondage number of Cartesian product of two paths P_n and P_m have been determined for $m \leq 4$. For more results on bondage number of a graph we suggest the survey [15].

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The following two lemmas show general bounds for the bondage number of a graph.

Lemma 1 (*[5]*). *If* u *and* v *are a pair of adjacent vertices of a graph* G*, then*

$$b(G) \leq d(u) + d(v) - 1 - |N(u) \cap N(v)|.$$

Lemma 2 (*[1,6]*). *If* u *and* v *are two vertices of a graph* G *such that* $d(u, v) \leq 2$*, then*

$$b(G) \leq d(u) + d(v) - 1.$$

2. Bondage number of $P_n \boxtimes P_m$

Let G and H be two graphs with the sets of vertices $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{u_1, u_2, \ldots, u_m\}$, respectively. The strong product of G and H is the graph $G \boxtimes H$ formed by the vertices $V = \{(v_i, u_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ and two vertices (v_i, u_j) and (v_k, u_l) are adjacent in $G \boxtimes H$ if and only if ($v_i = v_k$ and $u_j \sim u_l$), ($v_i \sim v_k$ and $u_j = u_l$) or ($v_i \sim v_k$ and $u_j \sim u_l$). In this section we will study the bondage number of the strong product of two paths P_n and P_m of order at least two. We begin by giving some observations and lemmas which will be useful into obtaining the bondage number of $P_n \boxtimes P_m$ for $n, m \geq 2$.

We will say that a graph G without isolated vertices satisfies the property $\mathscr{P}$ if it has a dominating set of minimum cardinality $S = \{u_1, u_2, \ldots, u_k\}$, $k = \gamma(G)$, such that $N[u_i] \cap N[u_j] = \emptyset$ for every $i, j \in \{1, \ldots, k\}$, $i \neq j$. Now, let $\mathfrak{F}$ be the class of all graphs satisfying property $\mathscr{P}$. Notice that for instance every path graph belongs to $\mathfrak{F}$.

Observation 1. *Let* $\{v_1, v_2, \ldots, v_n\}$ *be the set of vertices of a path* P_n *of order* n*. Then*

(i) *If* $n = 3t$*, then there is only one dominating set* S *of minimum cardinality in* P_n*; it satisfies property* $\mathscr{P}$ *and it is* $S = \{v_2, v_5, \ldots, v_{n-1}\}$.
(ii) *If* $n = 3t + 1$*, then there is only one dominating set* S *of minimum cardinality in* P_n *satisfying property* $\mathscr{P}$ *and it is* $S = \{v_1, v_4, v_7, \ldots, v_{n-3}, v_n\}$.
(iii) *If* $n = 3t + 2$*, then there are only two dominating sets* S *and* S' *of minimum cardinality in* P_n *satisfying property* $\mathscr{P}$ *and they are* $S = \{v_2, v_5, \ldots, v_{n-3}, v_n\}$ *and* $S' = \{v_1, v_4, v_7, \ldots, v_{n-1}\}$.

The following result from [16] is useful into studying the bondage number of $P_n \boxtimes P_m$.

Lemma 3 (*[16]*). *For any* $n, m \geq 2$,

$$\gamma(P_n \boxtimes P_m) = \gamma(P_n)\gamma(P_m) = \left\lceil \frac{n}{3} \right\rceil \left\lceil \frac{m}{3} \right\rceil.$$

Theorem 4. *For any* $n, m \geq 2$,

$$1 \leq b(P_n \boxtimes P_m) \leq 5.$$

Proof. Since $n, m \geq 2$, we have that there are always two adjacent vertices u, v in $P_n \boxtimes P_m$ such that $d(u) = 3$, $d(v) \leq 5$ and $|N(u) \cap N(v)| = 2$. So, the result follows by Lemma 1. ■

Similarly to the case of Cartesian product, hereafter we will study the bondage number of $P_n \boxtimes P_m$ by making some cases.

Theorem 5. *If* $(n = 3t$ *and* $m = 3r)$ *or* $(n = 3t$ *and* $m = 3r + 2)$*, then*

$$b(P_n \boxtimes P_m) = 1.$$

Proof. Notice that if $n = 3t$ and $m = 3r$, then by Observation 1(i) there exists only one dominating set of minimum cardinality in P_n and only one dominating set of minimum cardinality in P_m and they satisfy the property $\mathscr{P}$. Thus, there exists only one dominating set S, of minimum cardinality in $P_n \boxtimes P_m$; and it also satisfies the property $\mathscr{P}$. So, every vertex outside of S is dominated by only one vertex from S. Therefore, by deleting any edge e of $P_n \boxtimes P_m$ between a vertex of S and other vertex outside of S, we obtain that the domination number of $P_n \boxtimes P_m - \{e\}$ is greater than the domination number of $P_n \boxtimes P_m$.

On the other hand, let $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$ be the set of vertices of P_n and P_m, respectively. Since $n = 3t$, by Observation 1(i), we have that there is only one dominating set of minimum cardinality in P_n and it is $S_1 = \{u_2, u_5, \ldots, u_{n-1}\}$. Moreover, since $m = 3r + 2$ we have that every dominating set S_2 of minimum cardinality in P_m satisfies either

- $v_1 \in S_2$ and $v_2, v_3 \notin S_2$,
- or $v_2 \in S_2$ and $v_1, v_3 \notin S_2$.