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Coloring graphs without short cycles and long induced paths ☆

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ABSTRACT

For an integer $k \geq 1$, a graph G is k-colorable if there exists a mapping $c : V_G \to \{1, \dots, k\}$ such that $c(u) \neq c(v)$ whenever u and v are two adjacent vertices. For a fixed integer $k \geq 1$, the k-Coloring problem is that of testing whether a given graph is k-colorable. The girth of a graph G is the length of a shortest cycle in G. For any fixed $g \geq 4$ we determine a lower bound $\ell(g)$, such that every graph with girth at least g and with no induced path on $\ell(g)$ vertices is 3-colorable. We also show that for all fixed integers $k, \ell \geq 1$, the k-Coloring problem can be solved in polynomial time for graphs with no induced cycle on four vertices and no induced path on ℓ vertices. As a consequence, for all fixed integers $k, \ell \geq 1$ and $g \geq 5$, the k-Coloring problem can be solved in polynomial time for graphs with girth at least g and with no induced path on ℓ vertices. This result is best possible, as we prove the existence of an integer ℓ^*, such that already 4-Coloring is NP-complete for graphs with girth 4 and with no induced path on ℓ^* vertices.

1. Introduction

Graph coloring involves the labeling of the vertices of some given graph by k integers called colors such that no two adjacent vertices receive the same color. Due to the fact that the corresponding decision problem k-Coloring is NP-complete for any fixed $k \geq 3$, there has been considerable interest in studying its complexity when restricted to certain graph classes, see e.g. the surveys of Randerath and Schiermeyer [32] and Tuza [36]. We focus on graph classes defined by forbidden induced subgraphs. Before we summarize the known results and explain our new results, we first state the necessary terminology and notations.

1.1. Terminology

We only consider finite undirected graphs with no loops and no multiple edges. We refer to the textbook by Bondy and Murty [3] for any undefined graph terminology. Let $G = (V, E)$ be a graph. We write $G[U]$ to denote the subgraph of G *induced by* the vertices in U, that is, the subgraph of G with vertex set U and an edge between two vertices $u, v \in U$ whenever $uv \in E$. The *length* of a path or cycle is the number of its edges. The graphs C_n and P_n denote the cycle and path on n vertices, respectively. The graph $K_{r,s}$ denotes the complete bipartite graph with partition classes of size r and s, respectively. The disjoint union of two graphs G and H is denoted $G + H$, and the disjoint union of r copies of G is denoted rG. A *linear forest* is the disjoint union of a collection of paths. Let G be a graph and $\{H_1, \dots, H_p\}$ be a set of graphs. We say that G is

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$(H_1, \ldots, H_p)$-*free* if G has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; if $p = 1$, we sometimes write H_1-free instead of (H_1)-free. If G is C_3-free, then we also say that G is *triangle-free*. The *girth* $g(G)$ of G is the length of a shortest cycle in G. Note that G has girth at least p for some integer $p \geq 4$ if and only if G is $(C_3, \ldots, C_{p-1})$-free.

A *k-coloring* of a graph $G = (V, E)$ is a mapping $\phi : V \rightarrow \{1, \ldots, k\}$ such that $\phi(u) \neq \phi(v)$ whenever $uv \in E$. We say that $\phi(u)$ is the *color* of u. If G has a k-coloring, then G is said to be *k-colorable*. The *chromatic number* $\chi(G)$ of G is the smallest k such that G is k-colorable. If $\chi(G) = k$, then we also say that G is *k-chromatic*. The Coloring problem is that of testing whether a given graph admits a k-coloring for some given integer k. If k is *fixed*, that is, not part of the input, then we denote this problem as k-Coloring. The problem k-Precoloring Extension is that of deciding whether a given mapping $\phi_W : W \rightarrow \{1, \ldots, k\}$ defined on a (possibly empty) subset $W \subseteq V$ of a graph $G = (V, E)$ can be extended to a k-coloring of G.

1.2. Related work

Král' et al. [22] completely determined the computational complexity of Coloring for graph classes characterized by one forbidden induced subgraph H. They showed that Coloring can be solved in polynomial time for H-free graphs if H is an induced subgraph of P_4 or of $P_1 + P_3$, and that this problem is NP-complete if H is any other graph.

The computational complexity of the Coloring problem for (H_1, H_2)-free graphs where H_1 and H_2 are two distinct graphs is still open, although several partial results are known. In particular (C_3, H)-free graphs, or equivalently, H-free graphs with girth at least 4, are well studied. Král' et al. [22] showed that for any graph H that contains at least one cycle, 3-Coloring, and hence Coloring, is NP-complete for (C_3, H)-free graphs. Their work was extended by Schindl [34]. Maffray and Preissmann [28] showed that 3-Coloring and consequently, Coloring is NP-complete for $(C_3, K_{1,5})$-free graphs. Broersma et al. [8] showed that Coloring is polynomial-time solvable for $(C_3, 2P_3)$-free graphs, thereby completing a study of Dabrowski et al. [11], who considered the Coloring problem restricted to (C_3, H)-free graphs for graphs H with $|V_H| \leq 6$.

The computational complexity classification of k-Coloring for H-free graphs where k is a fixed integer and H is a fixed graph is also still open, but the following is known. Král' et al. [22] showed that 3-Coloring is NP-complete for graphs of girth at least g for any fixed $g \geq 3$. Kamiński and Lozin [21] used a similar reduction to show that k-Coloring is NP-complete for graphs of girth at least g for any fixed $k \geq 3$ and $g \geq 3$. Hence, for all $k \geq 3$, k-Coloring is NP-complete for H-free graphs if H contains a cycle. Holyer [18] showed that 3-Coloring is NP-complete for line graphs, whereas Leven and Galil [25] showed that k-Coloring is also NP-complete on line graphs for $k \geq 4$. Because every line graph is claw-free, that is, has no induced $K_{1,3}$, this means that for all $k \geq 3$, k-Coloring is NP-complete for H-free graphs if H is a forest that contains a vertex with degree at least 3. Hence, only the case in which H is a linear forest remains. Huang [19] proved that 4-Coloring is NP-complete for P_7-free graphs and that 5-Coloring is NP-complete for P_6-free graphs. In contrast to these hardness results, Couturier et al. [10] generalized a result for P_5-free graphs of Hoàng et al. [17] by proving that for any fixed integers k and r, the k-Coloring problem can be solved in polynomial time for $(P_5 + rP_1)$-free graphs, whereas Randerath and Schiermeyer [31] showed that 3-Coloring can be solved in polynomial time for P_6-free graphs. Broersma et al. [7] extended the latter result by showing that 3-Coloring is polynomial-time solvable for H-free graphs if H is a linear forest with $|V_H| \leq 6$ or $H = rP_3$ for any integer r. Moreover, it is known that 4-Coloring is polynomial-time solvable for $(P_2 + P_3)$-free graphs [14].

The k-Coloring problem has also been studied for (H_1, H_2)-free graphs where H_1 and H_2 are two distinct graphs. We refer to Randerath and Schiermeyer [32] for a detailed survey on so-called good Vizing-pairs (A, B) that satisfy the condition that every (A, B)-free graph is 3-colorable, in particular when $A = C_3$. Brandt [4] showed that every (C_3, sP_2)-free graph is $(2s - 2)$-colorable for any $s \geq 3$.

1.3. Our results

We consider the relation between the girth of a graph and the length of a forbidden induced path for the k-Coloring problem. As a start, note that graphs with girth $g = \infty$ are forests, and consequently, these graphs are 2-colorable. What if g is finite? In Section 2 we determine, for any fixed girth $g \geq 4$, a lower bound $\ell(g)$ such that every $P_{\ell(g)}$-free graph with girth at least g is 3-colorable. This extends the result of Sumner [35] who showed that every P_5-free graph of girth at least 4 is 3-colorable complementing a result of Randerath and Schiermeyer [32], who showed that for all $\ell \geq 4$, every P_ℓ-free graph of girth at least 4 is $(\ell - 2)$-colorable. Our results lead to Table 1. Note that for the cases $g \in \{4, 5, 7\}$ the lower bounds are slightly worse than the lower bound for the general case $g \geq 8$; the difference between them is 1. The proofs of the results in Table 1 are constructive, that is, they yield polynomial-time algorithms for solving 3-Coloring on these graph classes.

In Section 3.1 we show that for all integers $k, \ell, r, s \geq 1$, the k-Coloring problem is polynomial-time solvable on $(K_{r,s}, P_\ell)$-free graphs by using a recent result of Atminas, Lozin and Razgon [1]. By taking $r = s = 2$, we obtain that for all integers $k, \ell \geq 1$, the k-Coloring problem is polynomial-time solvable on (C_4, P_ℓ)-free graphs. Consequently, for all integers $g \geq 5$ and $k, \ell \geq 1$, the k-Coloring problem is polynomial-time solvable on P_ℓ-free graphs of girth at least g. As every graph has girth at least 3, and 3-Coloring is NP-complete in general, the case $g = 4$ remains. We solve this case in Section 3.2 by showing that even 4-Coloring is NP-complete for (C_3, P_{164})-free graphs, that is, for P_{164}-free graphs of girth at least $g = 4$. This is a new result as all the gadgets used in the proofs of the aforementioned NP-completeness results on k-Coloring for P_ℓ-free graphs are not triangle-free, that is, have girth equal to 3. We expect that $\ell = 164$ can be improved,