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A formulation of the wide partition conjecture using the atom problem in discrete tomography

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ABSTRACT

The Wide Partition Conjecture (WPC) was introduced by Chow and Taylor as an attempt to prove inductively Rota's Basis Conjecture, and in the simplest case tries to characterize partitions whose Young diagram admits a "Latin" filling. Chow et al. (2003) showed how the WPC is related to problems such as edge-list coloring and multi-commodity flow. As far as we know, the conjecture remains widely open.

We show that the WPC can be formulated using the k-atom problem in Discrete Tomography, introduced in Gardner et al. (2000). In this approach, the WPC states that the sequences arising from partitions admit disjoint realizations if and only if any combination of them can be realized independently. This realizability condition can be checked in polynomial time, although is not sufficient in general Chen and Shastri (1989), Guíñez et al. (2011). In fact, the problem is NP-hard for any fixed $k \geqslant 2$ Dürr et al. (2012). A stronger condition, called the saturation condition, was introduced in Guíñez et al. (2011) to solve instances where the realizability condition fails. We prove that in our case, the saturation condition is implied by the realizability condition. Moreover, we show that the saturation condition can be obtained as the Lagrangian dual of the linear programming relaxation of a natural integer programming formulation of the k-atom problem.

1. Introduction

A partition is a sequence of integers $\lambda = (\lambda_1, \dots, \lambda_\ell)$ ordered as $\lambda_1 \geqslant \cdots \geqslant \lambda_\ell > 0$. It can be described through the Young tableau Y_λ, which is a collection of cells arranged in left justified rows, with λ_i cells in row i. We say that a partition λ is *Latin* if the cells of Y_λ can be labeled such that row i contains the integers $1, \dots, \lambda_i$ and such that no two squares in the same row or column have the same value. See Fig. 1 for an example.

The *conjugate* of λ is the partition λ^* with $\lambda^*_j = |\{i : \lambda_i \geqslant j\}|$. Here Y_{λ^*} is obtained from Y_λ by interchanging rows and columns. A subpartition μ of λ, denoted $\mu \subseteq \lambda$, is a partition obtained by deleting some parts of λ. Equivalently, Y_μ is obtained from Y_λ by deleting some rows and making the remaining rows adjacent.

We say that $\lambda = (\lambda_1, \dots, \lambda_\ell)$ *dominates* $\mu = (\mu_1, \dots, \mu_k)$, written $\lambda \succcurlyeq \mu$, if $\sum_{i=1}^{j} \lambda_i \geqslant \sum_{i=1}^{j} \mu_i$ for each $j = 1, \dots, \ell$, and $\sum_{i=1}^{\ell} \lambda_i = \sum_{i=1}^{k} \mu_i$. A partition λ is *wide* if $\mu \succcurlyeq \mu^*$ for each $\mu \subseteq \lambda$.

As noticed in [3], every Latin partition is wide. The Wide Partition Conjecture states that this necessary condition is also sufficient.

Conjecture 1 (*WPC, See [3]*). *λ is Latin if and only if it is wide.*

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5	4	3	1	2
3	1	2		
2	3	1		
1	2			

Fig. 1. The tableau $Y_{(5,3,3,2)}$ is Latin.

The WPC was introduced by Chow and Taylor and originally motivated as an attempt to prove Rota's basis conjecture [14,3]. Chow et al. [3] showed how the WPC is related to some problems such as edge-list coloring, multi-commodity network flows and the Greene–Kleitman theorem. They also mentioned some connections with the invariant theory, although they did not make it explicit.

Let us denote $[n] = \{1, \ldots, n\}$ for convenience. The row projection of a subset M of the grid $[m] \times [n]$ is the vector $r \in \mathbb{Z}_+^m$ such that $r_i = |\{j : (i, j) \in M \text{ for some } j\}|$. The column projection $s \in \mathbb{Z}_+^n$ is defined analogously. Given a pair $b = (r, s) \in \mathbb{Z}_+^m \oplus \mathbb{Z}_+^n$, any subset M having r and s as row and column projection is a *realization* of b. If b admits a realization then we say that b is *realizable*. Notice that using the analogy between the $m \times n$ grid and the complete bipartite graph $K_{m,n}$, realizations of b correspond to b-factors of $K_{m,n}$ (see Section 3).

The k-atom problem consists in, given sequences $b^i = (r^i, s^i)$, $i = 1, \ldots, k$, finding pairwise disjoint realizations $M^1, \ldots, M^k$ of $b^1, \ldots, b^k$, respectively. If such realizations exist then we say that $(M^1, \ldots, M^k)$ is a $(b^1, \ldots, b^k)$-*packing*. This problem is motivated by the reconstruction problem of a polyatomic structure organized on a grid, where b^i are the projections of the atoms of type i. Gardner et al. [10] introduced it as a generalization of the binary matrix reconstruction problem under given row and columns sums, first studied by Ryser [16]. Many problems in discrete tomography can be modeled using the k-atom problem [4,5,11,1,8], which makes it a central problem in the area; see [12,13] for the foundations and recent advances in discrete tomography. Moreover, the k-atom problem can be seen as a k-commodity flow problem over a bipartite directed network and also as a problem of finding a 3-way consistency table of size $m \times n \times (k + 1)$ with specified line sums (or 2-margins as they are known in the statistical context); see [6,7] for definitions and complexity results and [17] for a summary of necessary conditions for the existence of such tables.

A necessary condition for the existence of a $(b^1, \ldots, b^k)$-packing is that $b^J = (r^J, s^J)$ is individually realizable for each $J \subseteq [k]$, where $r^J = \sum_{j \in J} r^j$ and $s^J = \sum_{j \in J} s^j$. We refer to this as the *realizability condition*. This condition can be checked in polynomial time using Ryser's algorithm [16,11]. Although this condition ensures the existence of k-packings for some special instances, it does not suffice in general, even for $k = 2$ [2]. In fact, the k-atom problem is NP-hard for any fixed $k \geqslant 2$ [8].

In [11], a stronger necessary condition was introduced and proved to be sufficient for a family of instances of the 2-atom problem. For $A \subseteq [m] \times [n]$ and a given sequence $b = (r, s)$, let $\min_b(A) = \min\{|M \cap A| : M \text{ is a realization of } b\}$. Guíñez et al. [11] showed how to calculate $\min_b(A)$ in polynomial time for any set A using the minimum-weight max-flow algorithm. Observe that if $(M^1, \ldots, M^k)$ is a $(b^1, \ldots, b^k)$-packing then $\sum_{i=1}^k \min_{b^i}(A) \leqslant \sum_{i=1}^k |M^i \cap A| \leqslant |A|$. Then we say that $(b^1, \ldots, b^k)$ satisfy the *saturation condition* if b^i is realizable for each $i \in [k]$, and for every set $A \subseteq [m] \times [n]$

$$\min_{b^1}(A) + \cdots + \min_{b^k}(A) \leqslant |A|. \tag{1}$$

Notice that the saturation condition requires to check a number of inequalities which is exponential in the size of the grid. It is still unknown if we can check it for a fixed number k of sequences in polynomial time.

As first contribution, we present an equivalent formulation of the WPC using the k-atom problem, which might import new tools and ideas to solve it. This formulation is presented in Section 2, where we define an instance $b^\lambda = (b^1, \ldots, b^\ell)$ for each partition λ such that there exists a b^λ-packing of a given grid if and only if λ is Latin. We also show that the wideness of λ is equivalent to the realizability condition of b^λ. In Section 3 we show that the saturation condition is strictly stronger than the realizability condition for the general k-atom problem. The main result of this section is however that for instances arising from partitions, these two conditions are equivalent. Finally, in Section 4 we discuss how the saturation condition can be obtained as a combinatorial analog of the dual of the relaxation of a Linear Program formulation (P) of the k-atom problem. The following scheme outlines our main contributions.