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Cycles in cube-connected cycles graphs[☆]

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ABSTRACT

Let n be a positive integer with $n \geq 3$. The cube-connected cycles graph CCC_n has $n \times 2^n$ vertices, labeled $(l, \mathbf{x})$, where $0 \leq l \leq n-1$ and $\mathbf{x}$ is an n-bit binary string. Two vertices $(l, \mathbf{x})$ and $(l', \mathbf{y})$ are adjacent if and only if either $\mathbf{x} = \mathbf{y}$ and $|l - l'| = 1$, or $l = l'$ and $\mathbf{y} = (\mathbf{x})^l$. Let $L(n)$ denote the set of all possible lengths of cycles in CCC_n. In this paper, we prove that $L(n) = \{n\} \cup \{i \mid i \text{ is even}, 8 \leq i \leq n+5, \text{ and } i \neq 10\} \cup \{i \mid n+6 \leq i \leq n2^n\}$ if n is odd; $L(4) = \{4\} \cup \{i \mid i \text{ is even and } 8 \leq i \leq 64\}$; and $L(n) = \{n\} \cup \{i \mid i \text{ is even}, 8 \leq i \leq n2^n, \text{ and } i \neq 10\}$ if n is even and $n \geq 6$.

1. Introduction

For the graph notations and definitions, we follow [5]. The cube-connected cycles graph CCC_n was introduced by Preparata and Vuillemin in 1981 [6] as a good alternative to the hypercube, having a fixed degree equal to 3 while including a small diameter compared to its number of vertices. This graph has since been studied by many people who showed, in particular, that it also has good properties as far as communications are concerned. It was proved that CCC_n contains a Hamiltonian cycle, which is an interesting property to realize distributed algorithms (for further details, see [1–3,7]). However, its cycle structure is still not completely known with regards to whether it contains cycles of any given length. Germa, Heydemann, and Sotteau [4] studied the existence of cycles of all lengths in the cube-connected cycles graph and obtained the following theorem.

Theorem 1.1 ([4]). *The cube-connected cycles graph CCC_3 contains cycles of length 3 and all lengths between 8 and 24. CCC_4 contains cycles of length 4 and cycles of all even lengths between 8 and 64. For $n \geq 5$, CCC_n contains cycles of length n and of every even length between 8 and $n2^n$ except 10 and possibly $n2^n - 2$. Furthermore, for n odd, CCC_n contains cycles of every odd length between $n+6$ and $n2^n - n - 2$ and also cycles of length $n2^n - n + 2$.*

Let $L(n)$ denote the set of all possible lengths of cycles in CCC_n. We restate Theorem 1.1 as the following theorem.

Theorem 1.2. *Let n be a positive integer with $n \geq 3$.*

(1) $L(3) \supseteq \{3\} \cup \{i \mid 8 \leq i \leq 24\}$;
(2) $L(4) \supseteq \{4\} \cup \{i \mid i \text{ is even and } 8 \leq i \leq 64\}$;

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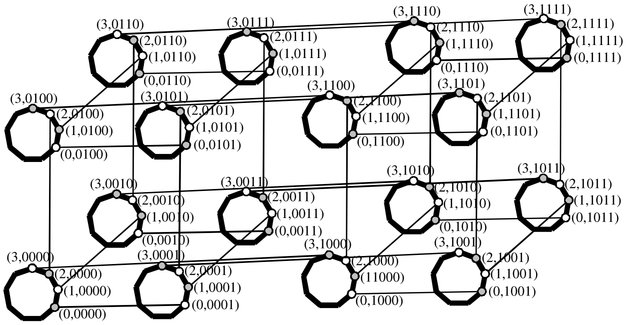

Fig. 1. CCC_4.

(3) $L(n) \supseteq \{n\} \cup \{i \mid i \text{ is even and } 8 \le i \le n+5\} \cup \{i \mid n+6 \le i \le n2^n\} - (\{10, n2^n-2, n2^n-n\} \cup \{n2^n - j \mid j \text{ is odd and } 1 \le j \le n-4\})$ *if* n *is odd and* $n \ge 5$; *and*
(4) $L(n) \supseteq \{n\} \cup \{i \mid i \text{ is even}, 8 \le i \le n2^n, \text{and } i \ne 10, n2^n-2\}$ *if* n *is even and* $n \ge 6$.

In this paper, we compute $L(n)$ for every integer n as follows:

Theorem 1.3. *Let* n *be a positive integer with* $n \ge 3$.

(1) $L(3) = \{3\} \cup \{i \mid 8 \le i \le 24\}$;
(2) $L(4) = \{4\} \cup \{i \mid i \text{ is even and } 8 \le i \le 64\}$;
(3) $L(n) = \{n\} \cup \{i \mid i \text{ is even}, 8 \le i \le n+5, \text{and } i \ne 10\} \cup \{i \mid n+6 \le i \le n2^n\}$ *if* n *is odd and* $n \ge 5$; *and*
(4) $L(n) = \{n\} \cup \{i \mid i \text{ is even}, 8 \le i \le n2^n, \text{and } i \ne 10\}$ *if* n *is even and* $n \ge 6$.

In the following section, we introduce the notation and the definition of cube-connected cycles graphs. In Section 3, we present some observations. With these observations, we can exclude certain integers l in $L(n)$. To prove Theorem 1.3, we only need to construct cycles of certain length in CCC_n. For this purpose, we first introduce some basic graphs of CCC_n in Section 4. Then we present four construction schemes for cycles of CCC_n in Sections 5–8. Finally, we prove Theorem 1.3 in Section 9.

2. Notations and definitions

Let $\mathbf{u} = u_{n-1}u_{n-2}\dots u_1u_0$ be an n-bit binary string. We use $(\mathbf{u})_j$ to denote the bit u_j and let $\mathbf{u} = (\mathbf{v})^i$ when $\mathbf{u}$ and $\mathbf{v}$ only differ in the bit at position i. Let $\mathbf{0}^n$ denote the n-bit binary string with $(\mathbf{0}^n)_i = 0$ for $0 \le i \le n-1$. The *Hamming weight* of $\mathbf{u}$, denoted by $w(\mathbf{u})$, is the number of i such that $(\mathbf{u})_i = 1$. Let $\mathbf{u} = u_{n-1}u_{n-2}\dots u_1u_0$ and $\mathbf{v} = v_{n-1}v_{n-2}\dots v_1v_0$ be two n-bit binary strings. The *Hamming distance* $h(\mathbf{u}, \mathbf{v})$ between two vertices $\mathbf{u}$ and $\mathbf{v}$ is the number of different bits in the corresponding strings of both vertices. The *n-dimensional hypercube*, denoted by Q_n, consists of all n-bit binary strings as its vertices and the two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent if and only if $h(\mathbf{u}, \mathbf{v}) = 1$. Obviously, Q_n is a bipartite graph with bipartition $A = \{\mathbf{u} \mid w(\mathbf{u}) \text{ is even}\}$ and $B = \{\mathbf{u} \mid w(\mathbf{u}) \text{ is odd}\}$.

The cube-connected cycles graph CCC_n has $n \times 2^n$ vertices, labeled $(l, \mathbf{x})$, where l is an integer between 0 and $n-1$ called the *level* of the vertex, and $\mathbf{x}$ is an n-bit binary string. Two vertices $(l, \mathbf{x})$ and $(l', \mathbf{y})$ are adjacent if and only if either $\mathbf{x} = \mathbf{y}$ and $|l - l'| = 1$ or $l = l'$ and $\mathbf{y} = (\mathbf{x})^l$. The edges joining two vertices of the same level are called *cube-edges*. To emphasize the level, the edge joining two vertices of level i is an *i-dimensional edge*. The edges that connect $(l, \mathbf{x})$ to its neighbors $(l+1, \mathbf{x})$ and $(l-1, \mathbf{x})$ are called *cycle-edges*. Moreover, these cycle-edges form a cycle of length n called the *fundamental cycle* defined by $\mathbf{x}$. For $n = 2$, CCC_n is simply the cycle of length 8. Therefore, in this article we will only consider the case $n \ge 3$. The cycle-edges for graph CCC_4 in Fig. 1 are indicated with dark lines.

Let $ab \in \{00, 01, 10, 11\}$. We set CCC_n^{ab} to denote the subgraph of CCC_n induced by the vertex set $\{(l, \mathbf{x}) \mid 0 \le l \le n-1, (\mathbf{x})_{n-1} = a \text{ and } (\mathbf{x})_{n-2} = b\}$. Obviously, $\deg_{CCC_n^{ab}}((n-1, \mathbf{x})) = \deg_{CCC_n^{ab}}((n-2, \mathbf{x})) = 2$ for any $\mathbf{x}$ with $(\mathbf{x})_{n-1} = a$ and $(\mathbf{x})_{n-2} = b$. We replace the path $\langle (n-3, \mathbf{x}), (n-2, \mathbf{x}), (n-1, \mathbf{x}), (0, \mathbf{x}) \rangle$ with edge $((n-3, \mathbf{x}), (0, \mathbf{x}))$ for any $\mathbf{x}$ with $(\mathbf{x})_{n-1} = a$ and $(\mathbf{x})_{n-2} = b$. The resultant graph is $(CCC_n^{ab})'$. It is easy to check that $(CCC_n^{ab})'$ is isomorphic to CCC_{n-2} for every $ab \in \{00, 01, 10, 11\}$.

With the above observation, we recursively construct CCC_n from four copies of CCC_{n-2} through the following three steps.

Step 1: let $ab \in \{00, 01, 10, 11\}$. We set $(CCC_n^{ab})'$ as the copy of CCC_{n-2} such that the vertex $(l, \mathbf{x})$ in $(CCC_n^{ab})'$ corresponds to the vertex $(l, \mathbf{y})$ in CCC_{n-2} satisfying $(\mathbf{x})_{n-1} = a$, $(\mathbf{x})_{n-2} = b$, and $(\mathbf{x})_i = (\mathbf{y})_i$ for $0 \le i \le n-3$. Note that $\mathbf{x}$ is an n-bit binary string and $\mathbf{y}$ is an $(n-2)$-bit binary string. In other words, two more bits ab are added to each vertex of CCC_{n-2}.

Step 2: for any n-bit binary string $\mathbf{x}$, insert all the $(n-2, \mathbf{x})$ and $(n-1, \mathbf{x})$ vertices. In other words, we replace the edge $((n-3, \mathbf{x}), (0, \mathbf{x}))$ with the path $\langle (n-3, \mathbf{x}), (n-2, \mathbf{x}), (n-1, \mathbf{x}), (0, \mathbf{x}) \rangle$. Obviously, the subgraph induced by those vertices with $(\mathbf{x})_{n-1} = a$ and $(\mathbf{x})_{n-2} = b$ is CCC_n^{ab}.