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Relationship between the edge-Wiener index and the Gutman index of a graph

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ABSTRACT

The Wiener index $W(G)$ of a connected graph G is defined to be the sum $\sum_{u,v} d(u,v)$ of the distances between the pairs of vertices in G. Similarly, the edge-Wiener index $W_e(G)$ of G is defined to be the sum $\sum_{e,f} d(e,f)$ of the distances between the pairs of edges in G, or equivalently, the Wiener index of the line graph $L(G)$. Finally, the Gutman index $\mathrm{Gut}(G)$ is defined to be the sum $\sum_{u,v} \deg(u)\deg(v)d(u,v)$, where $\deg(u)$ denotes the degree of a vertex u in G. In this paper we prove an inequality involving the edge-Wiener index and the Gutman index of a connected graph. In particular, we prove that $W_e(G) \geq \frac{1}{4}\mathrm{Gut}(G) - \frac{1}{4}|E(G)| + \frac{3}{4}\kappa_3(G) + 3\kappa_4(G)$ where $\kappa_m(G)$ denotes the number of all m-cliques in G. Moreover, the equality holds if and only if G is a tree or a complete graph. Using this result we show that $W_e(G) \geq \frac{\delta^2-1}{4}W(G)$ where δ denotes the minimum degree in G.

1. Introduction

For a graph G with vertex set $V = V(G)$ and edge set $E = E(G)$, let $\deg(u)$ and $d(u,v)$ denote the degree of a vertex $u \in V$ and the distance between vertices $u, v \in V$, respectively. Let $L(G)$ denote the line graph of G, that is, the graph with vertex set E and two distinct edges $e, f \in E$ adjacent in $L(G)$ whenever they share an endpoint in G. Furthermore, for $e, f \in E$, we let $d(e,f)$ denote the distance between e and f in the line graph $L(G)$. For adjacent vertices x and y, we write $x \sim y$.

In this paper we consider three important graph invariants, called *Wiener index* (denoted by $W(G)$ and introduced in [10]), *edge-Wiener index* (denoted by $W_e(G)$) and *Gutman index* (denoted by $\mathrm{Gut}(G)$), which are defined as follows:

$$W(G) = \sum_{\{u,v\}\subseteq V} d(u,v) = \frac{1}{2}\sum_{(u,v)\in V^2} d(u,v),$$

$$W_e(G) = \sum_{\{e,f\}\subseteq E} d(e,f) = \frac{1}{2}\sum_{(e,f)\in E^2} d(e,f),$$

$$\mathrm{Gut}(G) = \sum_{\{u,v\}\subseteq V} \deg(u)\ \deg(v)\, d(u,v) = \frac{1}{2}\sum_{(u,v)\in V^2} \deg(u)\ \deg(v)\, d(u,v).$$

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Note that edge-Wiener index of G is nothing but the Wiener index of the line graph $L(G)$ of G. Note also that in the literature a slightly different definition of the edge-Wiener index is sometimes used; for example, in [8] edge-Wiener index is defined to be $W_e(G) + \binom{n}{2}$ where $W_e(G)$ is as defined above and n is the order of G.

Besides applications in chemistry (see for example [7]), the Wiener index of a graph was studied also from a purely graph-theoretical point of view (for early results, see for example [6,9,3] for a nice survey). Generalizations of the Wiener index and relationship between these were studied in a number of papers (see for example [2,4,5,8]).

The main result of this paper is the following inequality, involving the edge-Wiener index and the Gutman index of a connected graph:

$$W_e(G) \geq \frac{1}{4}\mathrm{Gut}(G) - \frac{1}{4}|E(G)| + \frac{3}{4}\kappa_3(G) + 3\kappa_4(G), \tag{$*$}$$

where by $\kappa_m(G)$ we denote the number of m-cliques in G. In addition, we show that equality holds in $(*)$ if and only if G is a tree or a complete graph.

As a consequence of $(*)$, we prove the following inequality involving the Wiener index and the edge-Wiener index of a connected graph G:

$$W_e(G) \geq \frac{\delta^2 - 1}{4}\, W(G),$$

where $\delta = \delta(G)$ denotes the minimum degree in G. Notice that Wu [11] proved that for any graph G of minimum degree at least 2, $W_e(G) \geq W(G)$ with equality holding for cycles; see also [1] for the case when the minimum degree equals 2.

2. The proof

Throughout this section, let G be a connected graph with vertex set V and edge set E. Further, we let $A = \{(u, v) : uv \in E\}$ stand for the *arc set* of G. Recall that for any two edges $e = u_1u_2$ and $f = v_1v_2$ in E, the distance between e and f is defined as the distance $d_{L(G)}(e, f)$ between e and f in the line graph $L(G)$, and observe that if $e \neq f$, then

$$d(u_1u_2, v_1v_2) = \min\{d(u_i, v_j) : i, j \in \{1, 2\}\} + 1. \tag{1}$$

In addition to the distance between two edges, we will also consider the *average distance* between the endpoints of two edges, defined by

$$s(u_1u_2, v_1v_2) = \frac{1}{4}\big(d(u_1, v_1) + d(u_1, v_2) + d(u_2, v_1) + d(u_2, v_2)\big).$$

The average distance of endpoints has an interesting relationship with the Gutman index of a graph. Namely, if one wants to consider the version of the edge-Wiener index where the distances of edges in the sum are substituted by average distances of endpoints, then what one gets is essentially the Gutman index. More precisely, the following holds (which was also observed by Wu [11]):

Lemma 2.1. *Let G be a connected graph with vertex set V and edge set E. Then*

$$\frac{1}{2}\sum_{(e,f)\in E^2} s(e, f) = \frac{1}{4}\mathrm{Gut}(G).$$

Proof. Let A be the arc set of G. Then:

$$\frac{1}{2}\sum_{(e,f)\in E^2} s(e, f) = \frac{1}{8}\sum_{(u_1,u_2)\in A}\ \sum_{(v_1,v_2)\in A} \frac{1}{4}\big(d(u_1, v_1) + d(u_2, v_1) + d(u_1, v_2) + d(u_2, v_2)\big). \tag{$+$}$$

Now for each pair $i, j \in \{1, 2\}$, we see that

$$\begin{aligned}\sum_{(u_1,u_2)\in A}\ \sum_{(v_1,v_2)\in A} d(u_i, v_j) &= \sum_{u\in V}\ \sum_{u'\in N(u)}\ \sum_{v\in V}\ \sum_{v'\in N(v)} d(u, v)\\ &= \sum_{u\in V}\sum_{v\in V} \deg(u)\deg(v) d(u, v) = 2\,\mathrm{Gut}(G).\end{aligned}$$

By plugging this into $(+)$, we get

$$\frac{1}{2}\sum_{(e,f)\in E^2} s(e, f) = \frac{1}{8}\cdot\frac{1}{4}\cdot 4\cdot 2\cdot \mathrm{Gut}(G) = \frac{1}{4}\mathrm{Gut}(G),$$

as required. $\square$