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On packing colorings of distance graphs

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ABSTRACT

The *packing chromatic number* $\chi_\rho(G)$ of a graph G is the least integer k for which there exists a mapping f from $V(G)$ to $\{1, 2, \ldots, k\}$ such that any two vertices of color i are at a distance of at least $i+1$. This paper studies the packing chromatic number of infinite distance graphs $G(\mathbb{Z}, D)$, i.e. graphs with the set $\mathbb{Z}$ of integers as vertex set, with two distinct vertices $i, j \in \mathbb{Z}$ being adjacent if and only if $\mid i-j \mid \in D$. We present lower and upper bounds for $\chi_\rho(G(\mathbb{Z}, D))$, showing that for finite D, the packing chromatic number is finite. Our main result concerns distance graphs with $D = \{1, t\}$ for which we prove some upper bounds on their packing chromatic numbers, the smaller ones being for $t \geq 447$: $\chi_\rho(G(\mathbb{Z}, \{1, t\})) \leq 40$ if t is odd and $\chi_\rho(G(\mathbb{Z}, \{1, t\})) \leq 81$ if t is even.

1. Introduction

Let G be a connected graph and let k be an integer, $k \geq 1$. A *packing k-coloring* (or simply a packing coloring) of a graph G is a mapping f from $V(G)$ to $\{1, 2, \ldots, k\}$ such that for any two distinct vertices u and v, if $f(u) = f(v) = i$, then $\mathrm{dist}(u, v) > i$, where $\mathrm{dist}(u, v)$ is the distance between u and v in G (thus vertices of color i form an i-packing of G). The *packing chromatic number* $\chi_\rho(G)$ of G is the smallest integer k for which G has a packing k-coloring.

This parameter was introduced recently by Goddard et al. [9] under the name of *broadcast chromatic number* and the authors showed that deciding whether $\chi_\rho(G) \leq 4$ is NP-hard. Fiala and Golovach [6] showed that the packing coloring problem is NP-complete for trees. Brešar et al. [2] studied the problem on Cartesian products graphs, hexagonal lattice and trees, using the name of packing chromatic number. Other studies on this parameter mainly concern infinite graphs, with a natural question to be answered: *does a given infinite graph have finite packing chromatic number*? Goddard et al. answered this question affirmatively for the infinite two dimensional square grid by showing $9 \leq \chi_\rho \leq 23$. The lower bound was later improved to 10 by Fiala et al. [7] and then to 12 by Ekstein et al. [5]. The upper bound was recently improved by Holub and Soukal [13] to 17. Fiala et al. [7] showed that the infinite hexagonal grid has packing chromatic number 7; while both the infinite triangular lattice and the 3-dimensional square lattice were shown to admit no finite packing coloring by Finbow and Rall [8]. Infinite product graphs were considered by Fiala et al. [7] who showed that the product of a finite path (of order at least two) with the 2-dimensional square grid has infinite packing chromatic number while the product of the infinite path and any finite graph has finite packing chromatic number.

The (infinite) *distance graph* $G(\mathbb{Z}, D)$ with distance set $D = \{d_1, d_2, \ldots, d_k\}$, where d_i are positive integers, has the set $\mathbb{Z}$ of integers as vertex set, with two distinct vertices $i, j \in \mathbb{Z}$ being adjacent if and only if $|i-j| \in D$. The *finite distance graph* $G_n(D)$ is the subgraph of $G(\mathbb{Z}, D)$ induced by vertices $0, 1, \ldots, n-1$. To simplify, $G(\mathbb{Z}, \{d_1, d_2, \ldots, d_k\})$ will also be denoted as $D(d_1, d_2, \ldots, d_k)$ and $G_n(\{d_1, d_2, \ldots, d_k\})$ as $D_n(d_1, d_2, \ldots, d_k)$.

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Table 1
Lower and upper bounds for the packing chromatic number of $G(\mathbb{Z}, D)$ for different values of D. In the fourth column are the periods of the colorings giving the upper bounds.

D	$\chi_\rho \geq$	$\chi_\rho \leq$	Period
1, 2	8	8	54
1, 3	9[a]	9	32
1, 4	11	16	320
1, 5	10[a]	12	1028
1, 6	12	23	2016
1, 7	10[a]	15	640
1, 8	11[a]	25	5184
1, 9	10[a]	18	576
1, 2, 3	17	23	768
2, 3	11	13	240
2, 5	14	23	336

[a] Bound obtained by running Algorithm 1 of Section 4.

The study of distance graphs was initiated by Eggleton et al. [3]. A large amount of work has focused on colorings of distance graphs [4,15,1,11,12,14], but other parameters have also been studied on distance graphs, like the feedback vertex set problem [10].

The aim of this paper is to study the packing chromatic number of infinite distance graphs, with particular emphasis on the case $D = \{1, t\}$. In Section 2, we bound the packing chromatic number of the infinite path power (i.e. infinite distance graph with $D = \{1, 2, \ldots, t\}$). Section 3 concerns packing colorings of distance graphs with $D = \{1, t\}$, for which we prove some lower and upper bounds on the number of colors (see Proposition 1). Exact or sharp results for the packing chromatic number of some other 4-regular distance graphs are presented in Section 4. Section 5 concludes the paper with some remarks and open questions.

Our results about the packing chromatic number of $G(\mathbb{Z}, D)$ for some small values of D (from Sections 2 and 4) are summarized in Table 1.

The bounds of Section 3 are summarized in the following proposition:

Proposition 1. *Let t, q be integers. Then,*

$$\chi_\rho(D(1,t)) \leq \begin{cases} 89, & t = 2q+1,\ q \geq 35; \\ 40, & t = 2q+1,\ q \geq 223; \\ 179, & t = 2q,\ q \geq 89; \\ 81, & t = 2q,\ q \geq 224; \\ 29, & t = 96q \pm 1,\ q \geq 1; \\ 59, & t = 96q + 1 \pm 1,\ q \geq 1. \end{cases}$$

Some proofs of lower bounds use a density argument. For this, we define the density $\rho_a(G_n(D))$ of a color a in $G_n(D)$ as the maximum fraction of vertices colored a in any packing coloring of $G_n(D)$ and $\rho_a(D)$ (or simply ρ_a, if the graph is clear from the context) by $\rho_a(D) = \limsup_{n\to+\infty} \rho_a(G_n(D))$. Let also $\rho_{1,2}(G_n(D))$ be the maximum fraction of vertices colored 1 or 2 in any packing coloring of $G_n(D)$ and let $\rho_{1,2} = \limsup_{n\to+\infty} \rho_{1,2}(G_n(D))$. We have trivially, for any D, $\chi_\rho(G(\mathbb{Z}, D)) \geq \min\{c \mid \sum_{i=1}^{c} \rho_i \geq 1\}$ and $\rho_{1,2} \leq \rho_1 + \rho_2$.

2. Path Powers

The tth power G^t of a graph G is the graph with the same vertex set as G and edges between every vertices x, y that are at a mutual distance of at most t in G. Let $D^t = G(\mathbb{Z}, \{1, 2, \ldots, t\})$ be the tth power of the two-ways infinite path and let $P_n^t = G_n(\{1, 2, \ldots, t\})$ be the tth power of the path P_n on n vertices.

We first present an asymptotic result on the packing chromatic number:

Proposition 2. *$\chi_\rho(D^t) = (1 + o(1))3^t$ and $\chi_\rho(D^t) = \Omega(e^t)$.*

Proof. D^t is a spanning subgraph of the lexicographic product[1] $\mathbb{Z} \circ K_t$ (see Fig. 1). Then, as Goddard et al. [9] showed that $\chi_\rho(\mathbb{Z} \circ K_t) = (1 + o(1))3^t$, the same upper bound holds for D^t. To prove the lower bound, since $\rho_i \leq \frac{1}{it+1}$, then for any packing coloring of D^t using at most c colors, c must satisfy:

$$\sum_{i=1}^{c} \frac{1}{it+1} \geq 1.$$

[1] The lexicographic product $G \circ H$ of graphs G and H has vertex set $V(G) \times V(H)$ and two vertices (a, x) and (b, y) are linked by an edge if and only if $ab \in E(G)$ or $a = b$ and $xy \in E(H)$.