Discrete Applied Mathematics 167 (2014) 304–309

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The minimum restricted edge-connected graph and the minimum size of graphs with a given edge–degree☆

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ARTICLE INFO

Article history:
Received 26 January 2011
Received in revised form 21 October 2013
Accepted 24 October 2013
Available online 13 November 2013

Keywords:
Edge–degree
Extremal graph theory
Edge-connectivity
Restricted edge connectivity
Minimum restricted edge connected graphs

ABSTRACT

Let $G = (V(G), E(G))$ be a graph. Determining the minimum and/or maximum size ($|E(G)|$) of graphs with some given parameters is a classic extremal problem in graph theory. For a graph G and $e = uv \in E(G)$, we denote $d(e) = d(u) + d(v) - 2$ the edge–degree of e. In this paper, we obtain a lower bound for the minimum size of graphs with a given order n, a given minimum degree δ and a given minimum edge–degree $2\delta + k - 2$. Moreover, we characterize the extremal graphs for $k = 0, 1, 2$. As an application, we characterize some kinds of minimum restricted edge connected graphs.

1. The size of graph with given minimum degree and minimum edge degree

Let $G = (V(G), E(G))$ be a graph. Unless stated otherwise, we use [2] for terminology and notations not defined here. In particular, we call $|E(G)|$ the *size* of G. For $u \in V(G)$, the degree of u in G is denoted by $d_G(u)$, or simply $d(u)$. We denote $\delta(G)$ the minimum degree of G, $D_t(G)$ (or simply, D_t) the set of vertices of degree t in G, $D_{\geq t}(G)$ (or simply, $D_{\geq t}$) the set of vertices of degrees at least t in G, respectively. Let $d_t = |D_t(G)|$. For $e = uv \in E(G)$, denote by $d(e) = d(u) + d(v) - 2$ the *edge–degree* of e, and let $\xi(G) = \min\{d(e) : e \in E(G)\}$ (if e is a loop of G, then $u = v$). For $v \in V(G)$ and $A \subseteq V(G)$, we denote $N_G(v)$ the set of the neighbors of v in G and $N_G(A)$ the set $(\bigcup_{v\in V(A)} N_G(v)) \setminus V(A)$, respectively.

Determining the minimum and/or maximum size of graphs with some given parameters is a classic extremal problem in graph theory, see [1]. In this paper, we consider the following problem: What is the minimum size of graphs with a given order n, a given minimum degree δ and a given minimum edge–degree $2\delta+k-2$? We obtain a lower bound for the minimum size of graphs with a given order n, a given minimum degree δ and a given minimum edge–degree $2\delta + k - 2$. Moreover, we characterize the extremal graphs for $k = 0, 1, 2$. As an application, we characterize some kinds of minimum restricted edge connected graphs.

In the following theorem, the graph considered may have loops (a loop is an edge with two same endpoints). For a graph G and $u \in V(G)$, let $E_G(u)$ be the set of edges incident with u in G. When the graph G is understood from the context, we

☆ The research is supported by CSC and NSFC (No. 11301371 and No. 11171279).
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http://dx.doi.org/10.1016/j.dam.2013.10.028

write E_u for $E_G(u)$, δ and n for $\delta(G)$ and $|V(G)|$, respectively. Denote by $G_{n,\delta,k}$ a graph with order n, minimum degree δ, and minimum edge–degree $2\delta + k - 2$, and let $\mathcal{G}_{n,\delta,k}$ be the set of all $G_{n,\delta,k}$'s.

Theorem 1.1. *Let G be a graph with minimum degree $\delta \geq 3$, $\xi(G) \geq 2\delta + k - 2$. Then $|E(G)| \geq \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta$. Moreover, equality holds if and only if $d_\delta = \frac{\delta+k}{2\delta+k}n$ and $V(G) = D_\delta \cup N(D_\delta)$.*

Proof. Let $N(G) = N_G(D_\delta)$ and $T(G) = V \setminus (N \cup D_\delta)$ (or simply, we use N and T for $N(G)$ and $T(G)$). Note that $k \geq 0$ and the inequality holds if $k = 0$. Then we may assume that G is a graph with $\xi(G) \geq 2\delta - 1$, i.e. $k \geq 1$. Thus, D_δ is an independent set of G. The degrees of vertices in N are at least $\delta + k$ and the degrees of vertices in T are at least $\delta + 1$. We consider the size of G by distinguishing the following two cases.

If $|N| > \frac{\delta}{\delta+k}d_\delta$, we have

$$\begin{aligned}
|E(G)| &= \frac{\sum id_i}{2} \geq \frac{\delta d_\delta}{2} + \frac{\delta+k}{2}|N| + \frac{\delta+1}{2}|T| \\
&= \frac{\delta d_\delta}{2} + \frac{\delta+k}{2}|N| + \frac{\delta+1}{2}(n - d_\delta - |N|) \\
&= \frac{\delta}{2}n + \frac{n}{2} - \frac{d_\delta}{2} - \frac{\delta+1}{2}|N| + \frac{d_\delta+k}{2}|N| \\
&= \frac{\delta}{2}n + \frac{k-1}{2}|N| + \frac{\delta}{2}n - \frac{d_\delta}{2} \\
&> \frac{\delta}{2}n + \frac{k-1}{2}\left(\frac{\delta}{\delta+k}d_\delta\right) + \frac{\delta}{2}n - \frac{d_\delta}{2} \\
&= \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta - \frac{\delta}{2(\delta+k)}d_\delta + \frac{|N|+|T|}{2}. \qquad (1)
\end{aligned}$$

By the assumption, we have $|E(G)| \geq \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta$ since $\frac{|N|+|T|}{2} \geq \frac{\delta}{2(\delta+k)}d_\delta$.

If $|N| \leq \frac{\delta}{\delta+k}d_\delta$, we have

$$\begin{aligned}
|E(G)| &\geq \delta d_\delta + \frac{\delta+1}{2}|T| \\
&= \frac{\delta}{2}d_\delta + \frac{\delta}{2}\delta|N| + \frac{\delta}{2}d_\delta - \frac{\delta}{2}\delta|N| + \frac{\delta+1}{2}|T| \\
&= \frac{\delta}{2}n + \frac{\delta}{2}d_\delta - \frac{\delta}{2}|N| + \frac{1}{2}|T| \\
&\geq \frac{\delta}{2}n + \frac{\delta}{2}d_\delta - \frac{\delta}{2}\frac{\delta}{\delta+k}d_\delta + \frac{1}{2}|T| \\
&\geq \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta. \qquad (2)
\end{aligned}$$

By inequality (2), one can see that if $|N| < \frac{\delta}{\delta+k}d_\delta$, then $|E(G)| > \frac{\delta}{2}n + \frac{\delta k}{2(\delta+k)}d_\delta$. Moreover, if the equality holds in inequality (2), then $|T| = 0$. Note that $|N| = \frac{\delta}{\delta+k}d_\delta$ and $|T| = 0$ imply that $d_\delta = \frac{\delta+k}{2\delta+k}n$. Thus, the equality holds if and only if $d_\delta = \frac{\delta+k}{2\delta+k}n$ and $|N| = \frac{\delta}{2\delta+k}n$. We complete the proof. □

We next consider the extremal graphs for $k = 0, 1, 2$. By Theorem 1.1, we have $|E(G_{n,\delta,k})| \geq \lceil \frac{\delta}{2}n \rceil$ if $k = 0$. The following graph $H_{s,m} \in \mathcal{G}_{n,\delta,k}$ such that $|E(H_{s,m})| = \lceil \frac{\delta}{2}n \rceil$.

For integers $s, n, n \geq s + 1$, Harary [5] constructed classes of graphs $H_{s,n}$ that are minimum s-connected. The graph $H_{s,n}$ is as follows.

Case 1. $s = 2r, r > 0$. $H_{2r,n}$ is with vertex set $\{0, 1, 2, \ldots, n-1\}$ and two vertices i and j are adjacent if $i - r \leq j \leq i + r$, where addition is taken modulo n.

Case 2. $s = 2r + 1, r > 0$.

Case 2.1. n is even. Then $H_{2r+1,n}$ is obtained by adding edges joining vertex i to vertex $i + \frac{n}{2}$ for $1 \leq i < \frac{n}{2}$ on $H_{2r,n}$.

Case 2.2. n is odd. $H_{2r+1,n}$ is obtained by adding edges $[0, \frac{(n-1)}{2}]$ and $[0, \frac{(n+1)}{2}]$, and $[i, i+\frac{(n+1)}{2}]$ for $1 \leq i < \frac{(n-1)}{2}$ on $H_{2r,n}$.

Clearly, $H_{\delta,n}$ has minimum degree δ and the minimum edge–degree $2\delta+0-2$. In particular, $H_{\delta,n}$ has $\lceil \frac{\delta}{2}n \rceil$ edges. A graph G is called *almost n-regular* if there is at most one vertex of degree $n + 1$ and all other vertices have degree n. The following theorem follows immediately from the argument above.

Theorem 1.2. *$|E(G_{n,\delta,0})| \geq \lceil \frac{\delta}{2}n \rceil$ and $|E(G_{n,\delta,0})| = \lceil \frac{\delta}{2}n \rceil$ if and only if $G_{n,\delta,0}$ is an almost δ-regular graph.*