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Identifying codes and searching with balls in graphs

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ABSTRACT

Given a graph G and a positive integer R we address the following combinatorial search theoretic problem: What is the minimum number of queries of the form "does an unknown vertex $v \in V(G)$ belong to the ball of radius r around u?" with $u \in V(G)$ and $r \leq R$ that is needed to determine v. We consider both the adaptive case when the jth query might depend on the answers to the previous queries and the non-adaptive case when all queries must be made at once. We obtain bounds on the minimum number of queries for hypercubes, the Erdős–Rényi random graphs and graphs of bounded maximum degree.

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1. Introduction

Combinatorial search theory is concerned with problems of the following type "given a finite set S and an unknown element $x \in S$, find x as quickly as possible". In order to find x, we are allowed to ask questions of the type "is x contained in a subset $B \subseteq S$?" for various $B \subseteq S$. There are many real-world search problems which are of this type. For example given a set of blood samples, we might want to identify one which is infected. Or given a building, we might want to determine which room a person is hiding in.

Notice that to guarantee finding the unknown element x in a set S we always need to ask at least $\lceil \log_2 |S| \rceil$ questions. Indeed, to be able to distinguish between two elements $x, x' \in S$, we must at some point query a set $B \subseteq S$ such that $x \in B$ and $x' \notin B$. However if we ask less than $\lceil \log_2 |S| \rceil$ questions, then by the Pigeonhole Principle, there would be two elements x and $x' \in S$ which receive the same sequence of yes/no answers.

The lower bound $\lceil \log_2 |S| \rceil$ on the number of questions can be tight. In fact it is always possible to find x using $\lceil \log_2 |S| \rceil$ questions. Indeed first we divide S in half and ask which half x is in. Then we divide the half containing x in half again, and find out which one x is in. Proceeding this way, we reduce the subset of S which x may be in by half at every step, and hence we find x after at most $\lceil \log_2 |S| \rceil$ queries. This is the so called *halving search*.

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Notice that in the above search the set we query at each step depends on the answers we got in the previous steps. We say that a search is *adaptive* if a question is allowed to depend on answers to previous questions. In practice we often would like to make all our queries at the same time (for example if each individual query takes a long time, it would save time to make queries simultaneously). We say that a search is *non-adaptive* if the queries do not depend on answers obtained from previous ones. Just like in the adaptive case, there exist non-adaptive searches using the optimal $\lceil \log_2 |S| \rceil$ number of queries. For example if S is the set of 0–1 sequences of length n, then we can find $x = (x_1, \dots, x_n)$ using the $n = \lceil \log_2 |S| \rceil$ queries "is x_i equal to zero or one?"

Note that the answers to a set Q of query sets surely determine the unknown element x if and only if for any pair of elements $s_1, s_2 \in S$ there exists a query $B \in Q$ with $|B \cap \{s_1, s_2\}| = 1$. In this case we say that B *separates* s_1 and s_2, and Q is said to be a *separating system* or said to possess the *separating property*.

In this paper we study a search problem when the set S is a metric space, and we are only allowed to query sets B which are balls in the metric space. We call this the *ball search*. There are natural search problems of this type. For example if we want to track someone moving through a building, we might set up sensors at various points in the building which tell us if the person is within a certain range. This problem is inherently non-adaptive since the person is moving (and it would usually be inpractical to change the location of the sensors as the person moves). Another example of a ball search is to locate some object in the night sky using a telescope. This problem is inherently adaptive since once we capture the object with the telescope, we would want to zoom in order to get a better idea of its location.

Ball searches have been studied before, under the name of *identifying codes on graphs*. A ball $B_G(v, r)$ of radius r at vertex v in a graph G is the set of vertices in G that have distance at most r from v. (If G is specified, we simply write $B(v, r)$.) In general, if we consider a set X of vertices in G, $B_G(X, r)$ or simply $B(X, r)$ denotes the set of vertices in G that have distance at most r from X.

An $(r, \le l)$-*identifying code* of the graph G is a set C of vertices such that for every pair of distinct subsets $X, Y \subset V(G)$ with $0 < |X|, |Y| \le l$ the sets $B(X, r) \cap C$ and $B(Y, r) \cap C$ are different and non-empty. Notice that given a graph with an $(r, \le 1)$-identifying code C, we can always find an unknown vertex using the non-adaptive search querying the balls $\{B(c, r) : c \in C\}$. Identifying codes were introduced by Karpovsky, Chakrabarty, and Levitin [16] and the problem of determining the minimum size $i_r^{(l)}(G)$ of an identifying code in G has since attracted the attention of many researchers (for a full bibliography see [20]). An adaptive version of identifying codes was introduced by Moncel in his doctoral thesis [21] and in [3]. The minimum number of queries needed in the adaptive case is denoted by $a_r^{(l)}(G)$.

We will study a slight variant of identifying codes in which the restriction that the balls in the search must cover *all* the vertices of G is omitted. The reason we study this variant is that it is unnatural in a searching theory point of view and that it seems to have links to a natural combinatorial object (namely the Fano Plane. See Section 2 for details). We denote by $M(G, r)$ the minimum number of queries needed to non-adaptively find an unknown vertex $v \in G$ using balls of radius r. Similarly we denote by $A(G, r)$ the minimum number of queries needed to adaptively find an unknown vertex $v \in G$ using balls of radius r. Notice that we always have $i_r^{(1)}(G) \le M(G, r) \le i_r^{(1)}(G) + 1$ and $a_r^{(1)}(G) \le A(G, r) \le a_r^{(1)}(G) + 1$. Therefore, since we will mostly be interested in asymptotic estimates, it usually will not matter whether we study the quantity $M(G, r)$ or $i_r^{(1)}(G)$.

Notice that in the above ball searches we specified that we could only query balls of radius *exactly* r. With this restriction it is not always possible to find an unknown vertex v—for example if r is bigger than the diameter of G (the maximum distance between two vertices in G), then querying balls of radius r gives no information about the location of v. At the 5th Emléktábla Workshop, Gyula Katona asked what happens [18] if balls of radius *at most* r should be allowed as queries. We will denote by $M(G, \le r)$ and $A(G, \le r)$ the minimum number of queries to find an unknown vertex $v \in G$ using balls of radius at most r in the non-adaptive and adaptive searches respectively. We will denote by $M(G)$ and $A(G)$ the minimum number of queries to find an unknown vertex $v \in G$ using balls (of any radius) in the non-adaptive and adaptive searches respectively. Notice that since balls of radius zero are just single vertices in G, we always have $M(G, \le r), A(G, \le r) \le |G| - 1$. Also, we trivially have $M(G, \le r) \le M(G, r)$ and $A(G, \le r) \le A(G, r)$. The quantities $M(G, \le r)$ and $A(G, \le r)$ are monotonic in the sense that $M(G) \le M(G, \le r + 1) \le M(G, \le r)$ and $A(G) \le A(G, \le r + 1) \le A(G, \le r)$ hold for all r. Note that in any graph G two vertices $x, y \in V(G)$ are separated by a ball of radius r with center $z \in V(G)$ if and only if $z \in B_G(x, r) \triangle B_G(y, r)$, where $X \triangle Y = X \setminus Y \cup Y \setminus X$ for any two sets X and Y.

The first metric spaces in which we will study the ball search are hypercubes. Let Q_n be the n dimensional hypercube—the set of all 0–1 vectors of length n. We define a graph on Q_n by placing an edge between two vectors whenever they differ in exactly one entry. In this graph the distance $d(\underline{u}, \underline{v})$ between two vertices is exactly the *Hamming distance* between $\underline{u}$ and $\underline{v}$, i.e. the number of entries on which $\underline{u}$ and $\underline{v}$ differ. If $\underline{c}$ is an element of Q_n and r is a non-negative integer, then $B(\underline{c}, r)$ denotes the ball of center $\underline{c}$ and radius r, that is, $B(\underline{c}, r) := \{\underline{v} : \underline{v} \in Q_n, d(\underline{c}, \underline{v}) \le r\}$. The most recent upper and lower bounds on $i_r^{(l)}(Q_n)$ were proved in [5,10,8,9,14]. Adaptive identification in Q_n was studied by Junnila [15] who obtained lower and upper bounds on $a_1^{(1)}(Q_n)$. To be able to state our results let $K(n, r)$ denote the minimum number of balls of radius r that cover all vertices of Q_n. The centers of such balls are said to form *covering codes* in Q_n [6]. It is known that for any fixed r, we have $K(n, r) = \Theta(2^n / V(n, r))$.

Theorem 1.1. *The functions $A(Q_n)$, $A(Q_n, \le r)$ and $A(Q_n, r)$ satisfy the following inequalities:*

(i)

$$n \le A(Q_n) \le n - 1 + \lceil \log(n + 1) \rceil .$$

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