Discrete Applied Mathematics 193 (2015) 48–56

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Chip-firing games on Eulerian digraphs and **NP**-hardness of computing the rank of a divisor on a graph

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ARTICLE INFO

Article history:
Received 25 July 2014
Received in revised form 20 February 2015
Accepted 28 April 2015
Available online 27 May 2015

Keywords:
Chip-firing game
Riemann–Roch theory
Computational complexity
Eulerian graph

ABSTRACT

Baker and Norine introduced a graph-theoretic analogue of the Riemann–Roch theory. A central notion in this theory is the rank of a divisor. In this paper we prove that computing the rank of a divisor on a graph is **NP**-hard, even for simple graphs.

The determination of the rank of a divisor can be translated to a question about a chip-firing game on the same underlying graph. We prove the **NP**-hardness of this question by relating chip-firing on directed and undirected graphs.

1. Introduction

The Riemann–Roch theory for graphs was introduced by Baker and Norine in 2007 as the discrete analogue of the Riemann–Roch theory for Riemann surfaces [4]. They defined the notions *divisor, linear equivalence* and *rank* also in this combinatorial setting, and showed that the analogue of basic theorems as for example the Riemann–Roch theorem, remains true. Theorems like Baker's specialization lemma [3] establish a connection between the rank of a divisor on a graph and on a curve, which enables a rich interaction of the discrete and continuous theories.

A central notion in the Riemann–Roch theory is the rank of a divisor. The question whether the rank can be computed in polynomial time has been posed in several papers [9,11,5], originally attributed to H. Lenstra.

Let us say a few words about previous work concerning the computation of the rank. Hladký, Král' and Norine [9] gave a finite algorithm for computing the rank of a divisor on a metric graph. Manjunath [11] gave an algorithm for computing the rank of a divisor on a graph (possibly with multiple edges), that runs in polynomial time if the number of vertices of the graph is a constant. It can be decided in polynomial time, whether the rank of a divisor on a graph is at least c, where c is a constant [5]. Computing the rank of a divisor on a complete graph can be done in polynomial time [8]. For divisors of degree greater than $2g-2$ (where g is the genus of the graph), the rank can be computed in polynomial time [11]. On the other hand, there is a generalized model in which deciding whether the rank of a divisor is at least zero is already **NP**-hard [1].

Our main goal in this paper is to show that computing the rank of a divisor on a graph is **NP**-hard, even for simple graphs. This result implies also the **NP**-hardness of computing the rank of a divisor on a tropical curve by [10, Theorem 1.6]. We also show that deciding whether the rank of a divisor on a graph is at most k is in **NP**.

Our method is the following: We translate the question of computing the rank of a divisor to a question about the chip-firing game of Björner, Lovász and Shor using the duality between these frameworks discovered by Baker and Norine [4].

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http://dx.doi.org/10.1016/j.dam.2015.04.030

We get that the following question is computationally equivalent to the determination of the rank: Given an initial chip-distribution on an (undirected) graph G, what is the minimum number of extra chips we need to put on this distribution to make the game non-terminating.

We first prove the **NP**-hardness of computing the minimum number of chips that enables a non-terminating game on a simple Eulerian digraph by showing that it equals to the number of arcs in a minimum cardinality feedback arc set. This result is mentioned in a note added in proof of [6], where only the larger or equal part is proved. Recently, Perrot and Pham [12] solved an analogous question in the abelian sandpile model, which is a closely related variant of the chip-firing game. Our result follows by applying their method to the chip-firing game.

Then we show that the second question (concerning chip-firing games on directed graphs) can be reduced to the first one (concerning undirected graphs). In order to do so, to any Eulerian digraph and initial chip-distribution, we assign an undirected graph with a chip-distribution such that in the short run, chip-firing on the undirected graph imitates chip-firing on the digraph.

2. Preliminaries

2.1. Basic notations

Throughout this paper, *graph* means a connected undirected graph that can have multiple edges but no loops. A graph is *simple* if it does not have multiple edges. A graph is usually denoted by G. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The degree of a vertex v is denoted by $d(v)$, the multiplicity of the edge (u, v) by $d(u, v)$. The *Laplacian matrix* of a graph G means the following matrix L:

$$L(i,j) = \begin{cases} -d(v_i) & \text{if } i = j \\ d(v_i, v_j) & \text{if } i \neq j. \end{cases}$$

Digraph means a (weakly) connected directed graph that can have multiple edges but no loops. We usually denote a digraph by D. The vertex set and edge set are denoted by $V(D)$ and $E(D)$, respectively. For a vertex v the indegree and the outdegree of v are denoted by $d^-(v)$ and $d^+(v)$, respectively. A digraph D is *Eulerian* if $d^+(v) = d^-(v)$ for each vertex $v \in V(D)$. The *head* of the directed edge $(u, v) \in E(D)$ is v, and the *tail* of the edge is u. The multiplicity of the directed edge (u, v) is denoted by $\overrightarrow{d}(u, v)$. A digraph is *simple* if $\overrightarrow{d}(u, v) \leq 1$ for each pair of different vertices $u, v \in V(D)$.

The Laplacian matrix of a digraph D means the following matrix L:

$$L(i,j) = \begin{cases} -d^+(v_i) & \text{if } i = j \\ \overrightarrow{d}(v_j, v_i) & \text{if } i \neq j. \end{cases}$$

An important notion concerning digraphs is the feedback arc set. It also plays a crucial role in this paper.

Definition 2.1. A *feedback arc set* of a digraph D is a set of edges $F \subseteq E(D)$ such that the digraph $D' = (V(D), E(D) \setminus F)$ is acyclic. We denote

$$\text{minfas}(D) = \min\{|F| : F \subseteq E(D) \text{ is a feedback arc set}\}.$$

Let G be a graph. An *orientation* of G is a directed graph D obtained from G by directing each edge. We identify the vertices of G with the corresponding vertices of D. We denote the indegree and the outdegree of a vertex $v \in V(G)$ in the orientation D by $d^-_D(v)$ and $d^+_D(v)$, respectively.

For a graph G let us denote by $\mathbf{0}_G$ the vector with each coordinate equal to 0, and by $\mathbf{1}_G$ the vector with each coordinate equal to 1, where the coordinates are indexed by the vertices of G. For a vertex v of G we denote the characteristic vector of v by $\mathbf{1}_v$. We use the same notations for digraphs.

2.2. Riemann–Roch theory on graphs

In this section we give some basic definitions of the Riemann–Roch theory on graphs. The basic objects are called *divisors*. For a graph G, $\text{Div}(G)$ is the free abelian group on the set of vertices of G. An element $f \in \text{Div}(G)$ is called *divisor*. We either think of a divisor $f \in \text{Div}(G)$ as a function $f : V(G) \to \mathbb{Z}$, or as a vector $f \in \mathbb{Z}^{|V(G)|}$, where the coordinates are indexed by the vertices of the graph.

The *degree* of a divisor is the following:

$$\deg(f) = \sum_{v \in V(G)} f(v).$$

The following equivalence relation on $\text{Div}(G)$ is called *linear equivalence*: For $f, g \in \text{Div}(G)$, $f \sim g$ if there exists a $z \in \mathbb{Z}^{|V(G)|}$ such that $g = f + Lz$.

A divisor $f \in \text{Div}(G)$ is *effective*, if $f(v) \geq 0$ for each $v \in V(G)$.