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Communication

A counterexample to the pseudo 2-factor isomorphic graph conjecture

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ABSTRACT

A graph G is *pseudo* 2-*factor isomorphic* if the parity of the number of cycles in a 2-factor is the same for all 2-factors of G. Abreu et al. conjectured that $K_{3,3}$, the Heawood graph and the Pappus graph are the only essentially 4-edge-connected pseudo 2-factor isomorphic cubic bipartite graphs (Abreu et al., 2008, Conjecture 3.6).

Using a computer search we show that this conjecture is false by constructing a counterexample with 30 vertices. We also show that this is the only counterexample up to at least 40 vertices.

A graph G is 2-*factor hamiltonian* if all 2-factors of G are hamiltonian cycles. Funk et al. conjectured that every 2-factor hamiltonian cubic bipartite graph can be obtained from $K_{3,3}$ and the Heawood graph by applying repeated star products (Funk et al., 2003, Conjecture 3.2). We verify that this conjecture holds up to at least 40 vertices.

1. Introduction and preliminaries

All graphs considered in this paper are simple and undirected. Let G be such a graph. We denote the vertex set of G by $V(G)$ and the edge set by $E(G)$.

A graph G is 2-*factor hamiltonian* if all 2-factors of G are hamiltonian cycles. Examples of 2-factor hamiltonian graphs include K_4, K_5, $K_{3,3}$ and the Heawood graph (see Fig. 1(a)). Funk et al. [7] have shown that 2-factor hamiltonian k-regular bipartite graphs only exist when $k \in \{2, 3\}$. They also constructed an infinite family of cubic bipartite 2-factor hamiltonian graphs obtained by applying repeated *star products* to $K_{3,3}$ and the Heawood graph.

Given two cubic graphs G_1, G_2. A graph G is a *star product* of G_1 and G_2 if and only if there is an $x \in V(G_1)$ with neighbors x_1, x_2, x_3 in G_1 and an $y \in V(G_2)$ with neighbors y_1, y_2, y_3 in G_2 such that $G = (G_1 - x) \cup (G_2 - y) \cup \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$.

Funk et al. conjectured that every 2-factor hamiltonian cubic bipartite graph belongs to their infinite family of 2-factor hamiltonian cubic bipartite graphs, i.e.:

Conjecture 1.1 (*Funk et al., Conjecture 3.2 in [7]*)**.** *Let G be a 2-factor hamiltonian k-regular bipartite graph. Then either $k = 2$ and G is a circuit or $k = 3$ and G can be obtained from $K_{3,3}$ and the Heawood graph by repeated star products.*

As already mentioned in [7], it follows from [9] that a smallest counterexample to this conjecture is cubic and cyclically 4-edge-connected and from [10] that it has girth at least 6. So to prove Conjecture 1.1, it would be sufficient to prove the following conjecture:

Conjecture 1.2 (*Funk et al. [7]*)**.** *The Heawood graph is the only 2-factor hamiltonian cyclically 4-edge-connected cubic bipartite graph of girth at least 6.*

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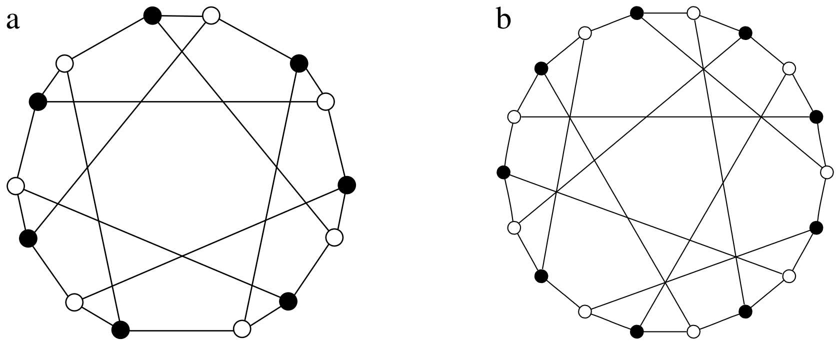

Fig. 1. The Heawood graph (a) and the Pappus graph (b).

Abreu et al. [1] extended these results on 2-factor hamiltonian graphs to the more general family of pseudo 2-factor isomorphic graphs. A graph G is *pseudo 2-factor isomorphic* if the parity of the number of cycles in a 2-factor is the same for all 2-factors of G.

Clearly all 2-factor hamiltonian graphs are also pseudo 2-factor isomorphic. An example of a pseudo 2-factor isomorphic graph which is not 2-factor hamiltonian is the Pappus graph (see Fig. 1(b)), as the cycle sizes of its 2-factors are (6,6,6) and (18) (see [1]).

Abreu et al. [1] proved that pseudo 2-factor isomorphic k-regular bipartite graphs only exist when $k \in \{2, 3\}$. They also constructed an infinite family of pseudo 2-factor isomorphic cubic bipartite graphs based on $K_{3,3}$, the Heawood graph and the Pappus graph and conjectured that these are the only pseudo 2-factor isomorphic cubic bipartite graphs:

Refuted Conjecture 1.3 (*Abreu et al., Conjecture 3.5 in [1]*). *Let G be a 3-edge-connected cubic bipartite graph. Then G is pseudo 2-factor isomorphic if and only if G can be obtained from $K_{3,3}$, the Heawood graph or the Pappus graph by repeated star products.*

Note that the pseudo 2-factor isomorphic graphs obtained by such a star product have a non-trivial 3-edge-cut. An edge cut E_0 of a graph G is *essential* if $G - E_0$ has at least two non-trivial components. A graph G is *essentially 4-edge-connected* if G does not have an essential edge cut E_0 with $|E_0| < 4$. Therefore Conjecture 1.3 can only hold if the following conjecture also holds:

Refuted Conjecture 1.4 (*Abreu et al., Conjecture 3.6 in [1]*). *Let G be an essentially 4-edge-connected pseudo 2-factor isomorphic cubic bipartite graph. Then G must be $K_{3,3}$, the Heawood graph or the Pappus graph.*

It follows from Theorem 3.10 in [1] that $K_{3,3}$ is the only essentially 4-edge-connected pseudo 2-factor isomorphic cubic bipartite graph with girth 4, so a counterexample to Conjecture 1.4 must have girth at least 6.

Abreu et al. partially proved this conjecture for *irreducible Levi graphs* (see [2] for details).

In the next section we describe the results of a computer search for cubic bipartite graphs of girth at least 6. This allowed us to verify Conjecture 1.2 up to 40 vertices. It also yielded one counterexample with 30 vertices to Conjecture 1.4. This is the only counterexample up to at least 40 vertices.

2. Testing and results

Using the program *minibaum* [3] we generated all cubic bipartite graphs with girth at least 6 up to 40 vertices and all cubic bipartite graphs with girth at least 8 up to 48 vertices. The counts of these graphs can be found in Table 1. Some of these graphs can be downloaded from http://hog.grinvin.org/Cubic.

We then implemented a program which tests if a given graph is pseudo 2-factor isomorphic and applied it to the generated cubic bipartite graphs. This yielded the following results:

Observation 2.1. *There is exactly one essentially 4-edge-connected pseudo 2-factor isomorphic graph different from the Heawood graph and the Pappus graph among the cubic bipartite graphs with girth at least 6 with at most 40 vertices.*

Observation 2.2. *There is no essentially 4-edge-connected pseudo 2-factor isomorphic graph among the cubic bipartite graphs with girth at least 8 with at most 48 vertices.*

This implies that Conjecture 1.4 (and consequently also Conjecture 1.3) is false. The counterexample has 30 vertices and there are no additional counterexamples up to at least 40 vertices and also no counterexamples among the cubic bipartite graphs with girth at least 8 up to at least 48 vertices. The counterexample (which we will denote by $\mathcal{G}$) is shown in Fig. 2 and its adjacency list can be found in Table 2. $\mathcal{G}$ can also be obtained from the *House of Graphs* [4] by searching for the keywords "pseudo 2-factor isomorphic * counterexample" where it can be downloaded and several of its invariants can be inspected.