Discrete Applied Mathematics 193 (2015) 80–93

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Restricted cycle factors and arc-decompositions of digraphs

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ARTICLE INFO

Article history:
Received 18 November 2014
Received in revised form 16 April 2015
Accepted 20 April 2015
Available online 16 May 2015

Keywords:
Cycle factor
2-factor
Mixed problem
NP-complete
Complexity
Cycle factors with no short cycles
Latin square
Avoidable arrays
Monochromatic matchings

ABSTRACT

We study the complexity of finding 2-factors with various restrictions as well as edge-decompositions in (the underlying graphs of) digraphs. In particular we show that it is $\mathcal{NP}$-complete to decide whether the underlying undirected graph of a digraph D has a 2-factor with cycles $C_1, C_2, \ldots, C_k$ such that at least one of the cycles C_i is a directed cycle in D (while the others may violate the orientation back in D). This solves an open problem from J. Bang-Jensen et al., Vertex-disjoint directed and undirected cycles in general digraphs, JCT B 106 (2014), 1–14. Our other main result is that it is also $\mathcal{NP}$-complete to decide whether a 2-edge-colored bipartite graph has two edge-disjoint perfect matchings such that one of these is monochromatic (while the other does not have to be). We also study the complexity of a number of related problems. In particular we prove that for every even $k \geq 2$, the problem of deciding whether a bipartite digraph of girth k has a k-cycle-free cycle factor is $\mathcal{NP}$-complete. Some of our reductions are based on connections to Latin squares and so-called avoidable arrays.

1. Introduction

Notation not introduced here follows [1,5]. We distinguish between *(non-directed) cycles* and *directed cycles* in digraphs, where the former is a subgraph that corresponds to a cycle in the underlying graph of a digraph. The notions of a *(non-directed) path* and a *directed path* are defined in a similar way. A *cycle factor* in a digraph is a spanning subgraph consisting of directed cycles, and a *2-factor* in a digraph (or graph) is a spanning subgraph consisting of cycles. We denote by $UG(D)$ the underlying graph of a directed graph D.

In this paper we consider several variations on the problem of finding cycle factors of digraphs. The problems of deciding if a given graph has a 2-factor and if a given digraph has a cycle factor are fundamental problems in combinatorial optimization, and both these problems are well-known to be solvable in polynomial time, see e.g. [1,5]. Here, we are particularly interested in problems concerning the complexity of deciding existence of spanning subgraphs that in a sense lies "in-between" 2-factors and cycle factors. In particular, we answer the question of the complexity of the following two problems by the first author:

Problem 1.1 (*[3, Problem 3]*). **2-factor with at least one directed cycle.**

Instance: A digraph D.
Question: Does D have a 2-factor F such that at least one cycle in F is a directed cycle, while the rest of the cycles do not have to respect the orientations of arcs in D?

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http://dx.doi.org/10.1016/j.dam.2015.04.020

Problem 1.2 ([13]). **Disjoint perfect matchings one of which is monochromatic**

Instance: A 2-edge colored bipartite graph $B = (U, V; E)$.
Question: Does B have two edge-disjoint perfect matchings M_1, M_2 so that every edge of M_1 has color 1, while M_2 may use edges of both colors?

This problem is equivalent to the following problem (see [1, Section 16.7]).

Problem 1.3. Semi-directed 2-factors of bipartite digraphs

Instance: A bipartite digraph $B = (X, Y; A)$.
Question: Does $UG(B)$ have a 2-factor which is the union of a perfect matching from X to Y in B (respecting the orientation) and a perfect matching in $UG(B)$?

Thus we are asking for a collection of cycles covering all vertices of B such that every second edge (starting from X) is oriented from X to Y in D, whereas the remaining edges do not have to respect the orientation of D.

The motivation for studying such "mixed" problems for digraphs, that is, problems concerning structures in a digraph D where only part of the structure has to respect the orientation of the arcs of D, is that this way one can obtain new insight into the complexity of various problems which have natural analogues for graphs and digraphs. As an example, in [2,3] the problem of deciding for a digraph D the existence of a directed cycle C in D and a cycle C' in $UG(D)$ which are vertex disjoint was studied. It was shown that this problem is polynomially decidable for the class of digraphs with a bounded number of cycle transversals of size 1 (vertices whose removal eliminates all directed cycles) and $\mathcal{NP}$-complete if we allow arbitrarily many transversal vertices. For (di)graphs one can decide the existence of two disjoint (directed) cycles in polynomial time [11,12].

Note that the variant of Problem 1.1 where we ask if D has a 2-factor F such that at most one cycle in F is not directed is $\mathcal{NP}$-complete. This can easily be proved as follows: it is $\mathcal{NP}$-complete to decide if a given graph G is Hamiltonian. Let D' be a digraph with a cycle factor, and let D'' be an acyclic orientation of the given graph G. Next, let D be the disjoint union of D' and D'' (or add an arbitrary arc between a vertex of D' and a vertex of D'' if one wants D to be connected). Then G is Hamiltonian if and only if D has a 2-factor where at most one cycle is not a directed cycle.

We show that Problems 1.1 and 1.2 are both $\mathcal{NP}$-complete.

Theorem 1.4. *2-factor with at least one directed cycle is $\mathcal{NP}$-complete.*

Theorem 1.5. *Disjoint perfect matchings one of which is monochromatic is $\mathcal{NP}$-complete.*

In fact, in the latter case, we shall prove that this problem is $\mathcal{NP}$-complete already for bipartite graphs with maximum degree 3.

Given an $n \times n$ array A where each cell contains a (possibly empty) subset of $\{1, 2, \dots, n\}$, we say that A is *avoidable* if there is an $n \times n$ Latin square L such that each cell of L does not contain a symbol that appears in the corresponding cell of A. We also say that *L avoids A*. If a cell in A is empty, then we also say that this cell contains entry $\emptyset$. In [6] it was proved that determining whether a given array where each cell contains a subset of $\{1, 2\}$ is avoidable is $\mathcal{NP}$-complete. Using Theorem 1.5 we can prove the following strengthening of that result.

Corollary 1.6. *The problem of determining whether an array where each cell contains either the set $\{1\}$, the set $\{1, 2\}$ or $\emptyset$ is avoidable is $\mathcal{NP}$-complete.*

Proof. We reduce Problem 1.2 to the problem of avoiding an array where each cell contains set $\{1\}$, the set $\{1, 2\}$ or is empty.

Let G be a balanced bipartite graph on $n + n$ vertices with an edge coloring f using colors 1 and 2. Let $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ be the parts of G. We form an $n \times n$ array A from G as follows:

- If $x_iy_j \notin E(G)$, then we set $A(i, j) = \{1, 2\}$.
- If $x_iy_j \in E(G)$ and $f(x_iy_j) = 2$, then we set $A(i, j) = \{1\}$.
- If $x_iy_j \in E(G)$ and $f(x_iy_j) = 1$, then set $A(i, j) = \emptyset$.

It is straightforward to verify that there are two disjoint generalized diagonals[1] D_1 and D_2 in A, where D_1 has no cell with symbol 1 and D_2 has no cell with symbol 2 (and therefore a Latin square avoiding A) if and only if G has two disjoint perfect matchings one of which does not contain any edge with color 2. □

Next, we consider a variant of a problem studied by Hartvigsen: in [10] he proved that the problem of deciding if a bipartite graph has a 4-cycle-free 2-factor (i.e. a 2-factor with no 4-cycle) can be solved in polynomial time. The problem of determining if a general graph has a 2-factor without 3-cycles is also solvable in polynomial time [9]. The analogous

[1] A generalized diagonal of an $n \times n$ matrix A is a collection of elements $a_{1,\pi(1)}, \dots, a_{n,\pi(n)}$ where π is a permutation of $[n]$.