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On coupon colorings of graphs

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ABSTRACT

Let G be a graph with no isolated vertices. A *k-coupon coloring* of G is an assignment of colors from $[k] := \{1, 2, \ldots, k\}$ to the vertices of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the *coupon coloring number* of G, and is denoted $\chi_c(G)$. In this paper, we prove that every d-regular graph G has $\chi_c(G) \geq (1 - o(1))d/\log d$ as $d \to \infty$, and the proportion of d-regular graphs G for which $\chi_c(G) \leq (1 + o(1))d/\log d$ tends to 1 as $|V(G)| \to \infty$.

An *injective k-coloring* of a graph G is an assignment of colors from $[k]$ to the vertices of G such that no two vertices joined by a path of length two in G have the same color. The minimum k for which such a coloring exists is called the *injective coloring number* of G, denoted $\chi_i(G)$. In this paper, we also discuss injective colorings of Hamming graphs.

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1. Introduction

Let G be a graph with no isolated vertices. A *k-coupon coloring* of G is an assignment of colors from $[k] := \{1, 2, \ldots, k\}$ to the vertices of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the *coupon coloring number* of G, and is denoted $\chi_c(G)$. This quantity is also referred to in Aram, Sheikholeslami and Volkmann [4] as the total domatic number of the graph. The coupon coloring number of any graph G with no isolated vertices is well-defined, since we may assign every vertex the same color. Note that in any coupon coloring of G, each color class must be a total dominating set of G.

The motivation for the term coupon coloring is that we may imagine the colors as coupons of different types, and then the requirement of coupon coloring is that every vertex collects from its neighbors coupons of all different types. If we imagine that users $v_1, v_2, \ldots, v_n$ are each assigned a bit from a k-bit message, and that every user has contact with a set of other users, then every user can reconstruct from her contacts the entire message if and only if the graph of contacts has a k-coupon coloring. The task given a graph of contacts is to determine the coupon coloring number of the graph, to maximize the length of the message that can be transmitted. It is possible to give examples of graphs of very large minimum degree whose coupon coloring number is 1 (see Section 1.1); for this reason, we consider d-regular graphs in this paper. The coupon coloring problem for hypercubes is closely related to problems in coding theory (see Östergård [23]), and for $k = 2$ the coupon coloring problem is also equivalent to the well-studied Property B of hypergraphs [13]: if we form from a graph G the *neighborhood hypergraph* $H = \{\Gamma(v) : v \in V(G)\}$ with vertex set $V(G)$, then a 2-coupon coloring of G exists if and only if H has Property B—namely, a 2-coloring of the vertices of H such that no edge of H is monochromatic. More generally,

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if H is a hypergraph and H admits a k-coloring such that every edge contains all k colors, then H is said to have a *panchromatic k-coloring*, which corresponds to a k-coupon coloring. For fixed k, local conditions on H were given by Kostochka and Woodall [19] for a hypergraph to have a panchromatic k-coloring. In particular it is shown that if every edge in H has size at least k, and for every set E of edges of H one has $|\bigcup E| \geq (k-1)|E| - k + 3$, then H has a panchromatic k-coloring. In this paper we study the extremal values of $\chi_c(G)$ when G is a d-regular graph as d becomes large.

In addition, results on coupon colorings have concrete applications in network science. One application is to large multi-robot networks (cf [7]). One may imagine a network large enough that robots must act based on local information. A graph can be constructed with robots in the network as nodes and an edge between nodes if the corresponding robots are able to communicate with each other. An example described in [1] is as follows: a group of robots is deployed to monitor an environment. Each robot must monitor many different statistics (e.g. temperature, humidity, etc.), but due to power limitations it is only equipped with a single sensor (thermometer, barometer, etc.). Thus, in order to obtain the remaining data, each robot must communicate with its neighbors. A similar example arises in allocating resources to a network [1]. If each vertex of a graph may only use resources available at the vertex or its neighbors, and if some resource (e.g. a printer) must be available to every node in the network, then copies of that resource must be allocated to a dominating set of the network. If every node in the network can accommodate one resource, then finding the coupon coloring number of the network is equivalent to finding the maximum number of resources that can be made available to every node in the network.

Finally, we remark that for real world applications it may be more appropriate to consider the closed neighborhood of a vertex, since a node in a network may use its own resource. However, we prefer to leave the neighborhood open in the definition of coupon coloring, as it is then more applicable to rainbow graphs, Property B, and panchromatic hypergraph coloring.

1.1. Coupon coloring regular graphs

For every positive integer d, there exists a graph of minimum degree at least d with coupon coloring number equal to 1: for instance this is easily verified for the bipartite graph of minimum degree d which represents the incidence graph of a complete d-uniform hypergraph on more than $2d-1$ vertices, since this hypergraph does not have Property B. Therefore $\chi_c(G)$ cannot be controlled by the minimum degree of a graph. The main theorem of this paper determines the asymptotic value of the coupon coloring number of almost all d-regular graphs, and also shows that every d-regular graph has coupon coloring number at least about $d/\log d$. This answers a question in Aram, Sheikholeslami and Volkmann [4], who showed $\chi_c(G) \geq d/3\log d$ for d-regular graphs G when $d \geq 3$. Let $G_{n,d}$ denote the probability space of d-regular n-vertex graphs, with the uniform probability measure.

Theorem 1.1. *For every $\delta > 0$, there exists $d_0(\delta)$ such that if $d \geq d_0(\delta)$, then every d-regular graph G has*

$$\chi_c(G) \geq (1-\delta)\frac{d}{\log d}.$$

For every $\epsilon > 0$, there exists $d_1(\epsilon)$ such that if $d \geq d_1(\epsilon)$, then as $n \to \infty$, almost every d-regular n-vertex graph G has

$$\chi_c(G) \leq (1+\epsilon)\frac{d}{\log d}.$$

If $G \in G_{n,d}$, then it was shown in [4] that $\chi_c(G) \leq d-1$ with probability tending to 1 as $n \to \infty$, which is not tight for large d. The second statement of the theorem also follows from the results of Alon [2] on total domination in random d-regular graphs; we give a short alternative proof of the statement. In particular, this theorem says $\chi_c(G) \sim d/(\log d)$ as $d \to \infty$ for almost every d-regular graph G. Theorem 1.1 is proved in Section 4. An explicit example of a regular graph with small coupon coloring number is the *Paley graph* G_q over $\mathbb{F}_q$ when $q \equiv 1 \bmod 4$ is a prime power. This graph is a $d = (q-1)/2$-regular graph with q vertices. The minimum size of a dominating set in G_q is known to be at least $(\frac{1}{2} - o(1))\log q$, using standard character sum estimates (see Gács and Szőnyi [16]), and since a coupon coloring requires every color class to be a total dominating set, $\chi_c(G_q) \leq (1+o(1))2q/\log q \leq (4+o(1))d/\log d$, which is within a factor four of the bound in Theorem 1.1. It would be interesting to provide explicit constructions of d-regular graphs with coupon coloring number $(1+o(1))d/\log d$, and in particular to determine $\chi_c(G_q)$ when G_q denotes the Paley graph over $\mathbb{F}_q$.

1.2. Injective and coupon colorings of cubes

We next consider graphs whose coupon coloring number is as large as possible. It is not hard to give examples of d-regular graphs G with $\chi_c(G) = d$, for instance, the complete bipartite graph with d vertices in each part has coupon coloring number equal to d. An *injective k-coloring* of a graph G is an assignment of colors from $[k]$ to the vertices of G such that no two vertices joined by a path of length two in G have the same color. Note that this is equivalent to saying that a color may appear at most once in the neighborhood of any vertex. The minimum k for which such a coloring exists is called the *injective coloring number* of G, denoted $\chi_i(G)$, and is well-defined since we may assign all vertices of G different colors. One observes

$$\chi_c(G) \leq \delta(G) \leq \Delta(G) \leq \chi_i(G),$$

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