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The matching energy of random graphs☆

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ABSTRACT

The matching energy of a graph was introduced by Gutman and Wagner, which is defined as the sum of the absolute values of the roots of the matching polynomial of the graph. For the random graph $G_{n,p}$ of order n with fixed probability $p \in (0, 1)$, Gutman and Wagner (2012) proposed a conjecture that the expectation of the matching energy of $G_{n,p}$ is asymptotically equal to $\frac{8\sqrt{p}}{3\pi}n^{\frac{3}{2}}$. In this paper, using analytical tools, we confirm this conjecture by obtaining a stronger result that the matching energy of $G_{n,p}$ is asymptotically almost surely equal to $\frac{8\sqrt{p}}{3\pi}n^{\frac{3}{2}}$.

1. Introduction

Let G be a finite simple graph of order n with vertex set $V(G)$ and edge set $E(G)$. A matching of G is a set of independent edges in G, and an r-matching of G is a matching of G that has exactly r edges. By $m_r(G)$ we denote the number of r-matchings in G. It is easy to verify that for $r < 0$ and $r > \lceil n/2 \rceil$, $m_r(G) = 0$. And when $r = 1$, $m_1(G)$ is the size of G. For convenience, we define $m_0(G) = 1$. The matching polynomial $m(G, x)$ [9,11,15] of a graph G is defined as

$$m(G,x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k m_k(G) x^{n-2k}.$$

The matching polynomial has been widely studied and many results on the properties of the matching roots have been obtained; see [9–12,15,18]. For any graph G, all the matching roots are real. If λ is a matching root, then $-\lambda$ is also a matching root. That is, the matching roots are symmetric. Moreover, the matching polynomial has many important implications in statistical physics and chemistry; see [13,16,18].

In [17], Gutman and Wagner introduced the matching energy (ME) of a graph G, which is defined as the sum of the absolute values of the roots of the matching polynomial of G. Note that the concept of the energy $E(G)$ of a simple undirected graph G was introduced by Gutman in [14]. Afterwards, there have been lots of research papers on this topic. A systematic study of this topic can be found in the book [20]. In [17], Gutman and Wagner pointed out that the matching energy is a quantity of relevance for chemical applications. Moreover, they arrived at the simple relation

$$TRE(G) = E(G) - ME(G),$$

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where $TRE(G)$ is the so-called "topological resonance energy" of the graph G. For more information about the applications of the matching energy, we refer the reader to [13,16]. Recently, there have been some results on the extremal values of the matching energy of graphs; see [5,19,21,22].

When we add an edge to a graph, the matching energy increases strictly.

Lemma 1.1 ([17]). *Let G be a graph and e one of its edges. Let $G-e$ be the subgraph obtained by deleting from G the edge e, but keeping all the vertices of G. Then*

$$ME(G-e) < ME(G).$$

Therefore, the complete graph K_n attains the maximum matching energy among all graphs of order n. In [17], Gutman and Wagner got the following lemma which gives an asymptotic estimation of the matching energy of K_n.

Lemma 1.2 ([17]). *The matching energy of the complete graph K_n is asymptotically equal to $\frac{8}{3\pi}n^{3/2}$. More precisely,*

$$ME(K_n) = \frac{8}{3\pi}n^{3/2} + O(n). \tag{1.1}$$

The above lemma can been thought as the upper bound of the matching energy of all graphs of order n. Moreover, they studied the lower bound of the matching energy of random graphs. Now we recall some notion in probability, we say an event holds almost surely (a.s.) if it occurs with probability 1. An event holds asymptotically almost surely (a.a.s.) if the probability of success goes to 1 as $n \to \infty$.

Lemma 1.3 ([17]). *Consider the random graph $G_{n,p}$ of order n with fixed probability $p \in (0, 1)$. Then*

$$ME(G_{n,p}) \geq \frac{\sqrt{p}}{\pi}n^{3/2} + O(\sqrt{n}\ln n) \tag{1.2}$$

holds asymptotically almost surely.

Based on the above analysis, they conjectured that

Conjecture 1.4 ([17]). *For any fixed probability $p \in (0, 1)$,*

$$n^{-3/2}E(ME(G_{n,p})) \longrightarrow \frac{8\sqrt{p}}{3\pi}$$

asymptotically almost surely, where $E(ME(G_{n,p}))$ is the expectation of $ME(G_{n,p})$.

This paper is to confirm the conjecture. The rest of the paper is organized as follows. In Section 2, we introduce the empirical matching distribution and list our main results: the empirical matching distribution converges weakly to the semicircle distribution; the asymptotic formula of the matching energy of random graphs. The explicit proofs will be shown in Sections 3 and 4. Throughout the paper we use the following standard asymptotic notation: as $n \to \infty$, $f(n) = o(g(n))$ means that $f(n)/g(n) \to 0$; $f(n) = O(g(n))$ means that there exists a constant C such that $|f(n)| \leq Cg(n)$.

2. Matching energy of random graphs

In this section, we present our main results of this paper.

Definition 2.1. For the random graph $G_{n,p}$ of order n with fixed probability $p \in (0, 1)$, let $x_1(G_{n,p}) \geq \cdots \geq x_n(G_{n,p})$ be the roots of the matching polynomial $m(G_{n,p}, x)$, since all roots of the matching polynomial are real. Then let $\lambda_i(G_{n,p}) = \frac{1}{\sqrt{np}}x_i(G_{n,p})$ for all $1 \leq i \leq n$.

We define the empirical matching distribution (EMD) as a distribution function $F_n(x)$ where

$$F_n(x) = \frac{1}{n}\Big|\{\lambda_i(G_{n,p})|\lambda_i(G_{n,p}) \leq x,\ i = 1, 2, \ldots, n\}\Big|.$$

The empirical matching distribution can be thought as the root distribution of the matching polynomial. Most work on the root distribution focuses on the spectral distributions of random matrices. The study can be traced back to the pioneer work-semicircle law discovered by Wigner in [25]. Afterwards, the research about the spectral distributions of many sorts of random matrices became the topics in mathematics and physics. For more details, we refer the reader to books [1,2,24]. In this paper, we find that the empirical matching distribution has the similar convergent property to the spectral distribution.