Discrete Applied Mathematics 193 (2015) 200–207

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Three-arc graphs: Characterization and domination

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ARTICLE INFO

Article history:
Received 25 February 2014
Received in revised form 12 April 2015
Accepted 21 April 2015
Available online 16 May 2015

Keywords:
3-arc graph
Domination number
Graph operation

ABSTRACT

An arc of a graph is an oriented edge and a 3-arc is a 4-tuple (v, u, x, y) of vertices such that both (v, u, x) and (u, x, y) are paths of length two. The 3-arc graph of a graph G is defined to have vertices the arcs of G such that two arcs uv, xy are adjacent if and only if (v, u, x, y) is a 3-arc of G. In this paper we give a characterization of 3-arc graphs and obtain sharp upper bounds on the domination number of the 3-arc graph of a graph G in terms that of G.

1. Introduction

The 3-arc construction [11] is a relatively new graph operation that has been used in the classification or characterization of several families of arc-transitive graphs [5,7,11,12,21,22]. (A graph is arc-transitive if its automorphism group acts transitively on the set of oriented edges.) As noted in [8], although this operation was first introduced in the context of graph symmetry, it is also of interest for general (not necessarily arc-transitive) graphs, and many problems on this new operation remain unexplored. In this paper we give partial solutions to two problems posed in [8] regarding 3-arc graphs.

An *arc* of a graph G is an ordered pair of adjacent vertices. For adjacent vertices u, v of G, we use uv to denote the arc from u to v, vu $(\neq uv)$ the arc from v to u, and $\{u, v\}$ the edge between u and v. A 3-*arc* of G is a 4-tuple (v, u, x, y) of vertices, possibly with $v = y$, such that both (v, u, x) and (u, x, y) are paths of G. Let Δ be a set of 3-arcs of G. Suppose that Δ is *self-paired* in the sense that $(y, x, u, v) \in \Delta$ whenever $(v, u, x, y) \in \Delta$. Then the 3-*arc graph of* G *relative to* Δ, denoted by $X(G, \Delta)$, is defined [11] to be the (undirected) graph whose vertex set is the set of arcs of G such that two vertices corresponding to arcs uv and xy are adjacent if and only if $(v, u, x, y) \in \Delta$. In the context of graph symmetry, Δ is usually a self-paired orbit on the set of 3-arcs under the action of an automorphism group of G. In the case where Δ is the set of all 3-arcs of G, we call $X(G, \Delta)$ the 3-*arc graph* [9] of G and denote it by $X(G)$.

The first study of 3-arc graphs of general graphs was conducted by Knor and Zhou in [9]. Among other things they proved that if G has vertex-connectivity $\kappa(G) \geq 3$ then its 3-arc graph has vertex-connectivity $\kappa(X(G)) \geq (\kappa(G) - 1)^2$, and if G is connected of minimum degree $\delta(G) \geq 3$ then the diameter $\mathrm{diam}(X(G))$ of $X(G)$ is equal to $\mathrm{diam}(G)$, $\mathrm{diam}(G) + 1$ or $\mathrm{diam}(G) + 2$. In [1], Balbuena, García-Vázquez and Montejano improved the bound on the vertex-connectivity by proving $\kappa(X(G)) \geq \min\{\kappa(G)(\delta(G) - 1), (\delta(G) - 1)^2\}$ for any connected graph G with $\delta(G) \geq 3$. They also proved [1] that for such a graph the edge-connectivity of $X(G)$ satisfies $\lambda(X(G)) \geq (\delta(G) - 1)^2$, and they further gave a lower bound on the restricted edge-connectivity of $X(G)$ in the case when G is 2-connected. In [8], Knor, Xu and Zhou studied the independence, domination and chromatic numbers of 3-arc graphs.

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http://dx.doi.org/10.1016/j.dam.2015.04.019

In a recent paper [20] we obtained a necessary and sufficient condition [20, Theorem 1] for $X(G)$ to be Hamiltonian. In particular, we proved [20, Theorem 2] that a 3-arc graph is Hamiltonian if and only if it is connected, and that if G is connected with $\delta(G) \geq 3$ then all its iterative 3-arc graphs $X^i(G)$ are Hamiltonian, $i \geq 1$. (The iterative 3-arc graphs are recursively defined by $X^1(G) := X(G)$ and $X^{i+1}(G) := X(X^i(G))$ for $i \geq 1$.) As a consequence we obtained [20, Corollary 2] that if a vertex-transitive graph is isomorphic to the 3-arc graph of a connected arc-transitive graph of degree at least three, then it is Hamiltonian. This provides new support to the well-known Lovász–Thomassen conjecture [17] which asserts that all connected vertex-transitive graphs, with finitely many exceptions, are Hamiltonian. We also proved (as a consequence of a more general result) [20, Theorem 4] that if a graph G with at least four vertices is Hamilton-connected, then so are its iterative 3-arc graphs $X^i(G)$, $i \geq 1$.

The 3-arc construction was generalized to directed graphs in [8]. Given a directed graph D, the *3-arc graph* [8] of D, denoted by $X(D)$, is defined to be the undirected graph whose vertex set is the set of arcs of D such that two vertices corresponding to arcs uv, xy of D are adjacent if and only if $v \neq x$, $y \neq u$ and u, x are adjacent in D. Recently, we proved with Wood [19] that the well-known Hadwiger's graph colouring conjecture [18] is true for the 3-arc graph of any directed graph with no loops.

In spite of the results above, compared with the well-known line graph operation [6] and the 2-path graph operation [3,10], our knowledge of 3-arc graphs is quite limited and many problems on them are yet to be explored. For instance, the following problems were posed in [8]:

Problem 1. Characterize 3-arc graphs of connected graphs.

Problem 2. Give a sharp upper bound on $\gamma(X(G))$ in terms of $\gamma(G)$ for any connected graph G with $\delta(G) \geq 2$, where γ denotes the domination number.

In this paper we give partial solutions to these problems. We first show that there is no forbidden subgraph characterization of 3-arc graphs (Proposition 1), and then we provide a descriptive characterization of 3-arc graphs (Theorem 2). We give a sharp upper bound for $\gamma(X(G))$ in terms of $\gamma(G)$ (Theorem 5) for any graph G with $\delta(G) \geq 2$, and more upper bounds for $\gamma(X(G))$ in terms of $\gamma(G)$ and the maximum degree $\Delta(G)$ when $2 \leq \delta(G) \leq 4$ (Theorem 6). Finally, we prove that if G is claw-free with $\delta(G) \geq 2$, then $\gamma(X(G)) \leq 4\gamma(G)$ and moreover this bound is sharp (Theorem 7).

All graphs in the paper are finite and undirected with no loops or multiple edges. The *order* of a graph is the number of vertices in the graph. As usual, the minimum and maximum degrees of a graph $G = (V(G), E(G))$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The degree of a vertex $v \in V(G)$ in G is denoted by $\deg(v)$. The *neighbourhood* of v in G, denoted by $N(v)$, is the set of vertices of G adjacent to v, and the *closed neighbourhood* of v is defined as $N[v] := N(v) \cup \{v\}$. We say that v *dominates* every vertex in $N(v)$, or every vertex in $N(v)$ is *dominated* by v. For a subset S of $V(G)$, denote $N(S) := \cup_{v \in S} N(v)$ and $N[S] := N(S) \cup S$. We may add subscript G to these notations (e.g. $\deg_G(v)$) to indicate the underlying graph when there is a risk of confusion. If $N[S] = V(G)$, then S is called a *dominating set* of G. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G; a dominating set of G with cardinality $\gamma(G)$ is called a $\gamma(G)$*-set* of G. The subgraph of G induced by S is denoted by $G[S]$, and the subgraph of G induced by $V(G) - S$ is denoted by $G - S$.

The reader is referred to [18] for undefined notation and terminology.

2. A characterization of 3-arc graphs

It is well known that line graphs can be characterized by a finite set of forbidden induced subgraphs [6]. In contrast, a similar characterization does not exist for 3-arcs graphs, as we show in the following result.

Proposition 1. *There is no characterization of 3-arc graphs by a finite set of forbidden induced subgraphs. More specifically, any graph is isomorphic to an induced subgraph of some 3-arc graph.*

Proof. Let H be any graph. Define H^* to be the graph obtained from H by adding a new vertex x and an edge joining x and each vertex of H. It is not hard see that $u, v \in V(H)$ are adjacent in H if and only if the arcs ux, vx of H^* are adjacent in $X(H^*)$. Thus the subgraph of $X(H^*)$ induced by $A := \{vx : v \in V(H)\} \subseteq V(X(H^*))$ is isomorphic to H via the bijection $v \leftrightarrow vx$ between $V(H)$ and A. Since H is arbitrary, this means that any graph is isomorphic to an induced subgraph of some 3-arc graph, and so the result follows. □

Next we give a descriptive characterization of 3-arc graphs. To avoid triviality we assume that the graph under consideration has at least one edge.

Theorem 2. *A graph G having at least one edge is isomorphic to the 3-arc graph of some graph if and only if $V(G)$ admits a partition $\mathcal{V} := \mathcal{V}_1 \cup \mathcal{V}_2$ and $E(G)$ admits a partition $\mathcal{E}$ such that the following holds:*

(a) *each element of $\mathcal{V}_1$ contains exactly one vertex of G, and each element of $\mathcal{V}_2$ is an independent set of G with at least two vertices;*
(b) *each $E_i \in \mathcal{E}$ induces a complete bipartite subgraph B_i of G with each part of the bipartition a subset of some $V \in \mathcal{V}_2$ with $|V| - 1$ vertices;*