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The maximum vertex coverage problem on bipartite graphs

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ABSTRACT

Given a simple undirected graph G and a positive integer s the Maximum Vertex Coverage Problem is the problem of finding a set U of s vertices of G such that the number of edges having at least one endpoint in U is as large as possible. We prove that the Maximum Vertex Coverage problem on bipartite graphs is NP-hard and discuss several consequences related to known combinatorial optimization problems.

1. Introduction

Preamble*. The story of the proof contained in this note dates back to* 2007. *For certain reasons (among them the length of the proof) we never published it before and ever since we have been waiting for a shorter proof that never came. Bruno passed away in* 2010. *Over time I replaced sadness by a sense of guilt for two reasons:* (1) *the proof has kept its length and* (2) *the proof has been still unpublished. As for* (1), *without Bruno, a shorter proof is even more unlikely to come. With this, I try to remedy to* (2) *at least.*

Given a simple undirected graph $G = (V, E)$ and a positive integer s, the Maximum Vertex Coverage Problem is the problem of finding a set U of s or fewer vertices such that the number of edges having at least one endpoint in U is as large as possible. The integer s, referred to as the *cardinality constraint*, is a part of the input.

The main aim of this work is to study the complexity of the Maximum Vertex Coverage Problem (MVC for shortness) restricted to bipartite graphs. Despite this strong restriction, MVC on bipartite graphs is still a basic interesting problem, because of its relations with other classical problems of combinatorial optimization: maximum coverage by cliques and maximum induced q-chromatic graph in special classes of perfect graphs, maximum s-packing of maximal matchings in bipartite graphs, bipartite heaviest and lightest subgraphs (see Section 4 for a details). Since all of these problems reduce to MVC on bipartite graphs we have a first motivation for studying its complexity.

A second motivation comes from the following straightforward remark: let **A** be any algorithm that solves MVC and let $G = (V, E)$ be any simple graph. Let U^*_s be the solution output by **A** with input G and s. If s^* is the least s such that the number of edges having at least one endpoint in $U^*_{s^*}$ equals $|E|$, then $U^*_{s^*}$ is a minimum cardinality vertex cover of G. Therefore, after linearly many calls to **A**, we solve the Vertex Cover problem (VC) on G. If follows that MVC is at least as hard as VC and since the latter problem is notoriously (and archetypally) NP-hard [24], we conclude that so is MVC. On the other hand, it is natural

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to wonder whether MVC is concretely harder than VC, that is, are there polynomial solvable classes of instances of the latter that are NP-hard for the former? To answer such a question let us consider the classes of line graphs, perfect graphs and bipartite graphs. Since the class of bipartite graphs is a proper subclass of the class of perfect graphs, let us consider line graphs and bipartite graphs only. Both classes form polynomial solvable classes of instances of VC [40]. By the results in [3,4], MVC is polynomial-time solvable on line graphs. Thus, if MVC is NP-hard on bipartite graphs then the question posed above has an affirmative answer.

MVC is itself a special case of Maximum Coverage problem. In the latter problem one is given an edge weighted hypergraph (V, E) (see Section 2 for definitions) and a nonnegative integer s and seeks a set U set of s or fewer vertices such that the total weight of the edges *covered* by U is as large as possible—with some abuse of terminology we say that a subset U of vertices of a (hyper)graph G covers edge e of G if $U \cap e \neq \emptyset$. Clearly, MVC is the restriction of Maximal Coverage to those hypergraphs whose edges have all the same weight and all have exactly two elements, that is to graphs.

The Maximum Coverage problem and, hence, MVC, has received a growing attention in the past decades from the viewpoint of approximability [1,9,14,15,23,29,30,33,38]. In contrast, little seems to be known about polynomial-time solvable instances of the problem. In the same way as MVC is at least as hard as VC, the Maximal Coverage problem is at least as hard as the HITTING SET problem (HS). The latter problem is the problem of finding a minimum cardinality or, more generally, a minimum weight *transversal* of a hypergraph, i.e. a subset of vertices covering all the edges. Clearly HS specializes to VC and therefore it is an NP-hard problem. Nonetheless nontrivial classes of polynomial-time solvable instances of HS are well-known and understood. Such classes form the subject of entire chapters of theoretical significance in textbooks in polyhedral combinatorics and hypergraph theory (see e.g., [5,13,18,28,36,37,39,40]). Most of the properties that imply the existence of a polynomial-time algorithm rely on the integrality properties of the following linear programming relation (i.e., of the related polyhedra and the corresponding defining systems of inequalities):

$$\max\{w^T y \mid y^T A(G) \leq c,\ y \in \mathbb{R}^E_+\} = \min\{c^T x \mid A(G)x \geq w,\ x \in \mathbb{R}^V_+\} \tag{1}$$

where $G = (V, E)$ is a hypergraph, $c \in \mathbb{Z}^V_+$, $w \in \mathbb{Z}^E_+$ are vectors and $A(G)$ is the edge–vertex incidence matrix of G. By specializing c and w in the relation above one generalizes the so-called König property for graphs in several directions. Recall that a graph G has König property when the maximum number of disjoint edges (the *matching number* $\nu(G)$) equals the minimum cardinality of a transversal (the *transversal number* $\tau(G)$). The reader is referred to Fig. 1 for some flavor, to Section 4 for some further background and to the aforementioned references for details. Roughly speaking, the hierarchy of Fig. 1 ranks hypergraphs by the integrality properties of (1): as we go up we solve more. Unimodular hypergraphs are topmost located in this hierarchy. Linear programming problems with totally unimodular matrix and integral data have integer optimal solutions. In particular, the polyhedra occurring in both the optimization problems in (1) are integral, i.e., their vertices (if any) have integer coordinates. Since bipartite graphs are unimodular hypergraphs, it follows that, if MVC is NP-hard in the class of bipartite graphs, then maximal coverage is NP-hard in the class of unimodular hypergraphs and, hence, in each superclass of unimodular hypergraph. It is worth observing that within graphs the classes of ideal, normal and unimodular hypergraphs, coincide. This discussion provides us with further motivation for studying the complexity of MVC in bipartite graphs.

In this paper we prove the following.

Theorem 1. *The Maximum Vertex Coverage problem is NP-Hard when restricted to bipartite graphs.*

Our previous discussion immediately implies the following consequences.

Corollary 1. *The Maximum Vertex Coverage problem is NP-Hard when restricted to perfect graphs.*

Corollary 2. *The Maximal Coverage problem is NP-Hard in the class of unimodular hypergraphs and, therefore, in the classes of balanced, Mengerian, ideal and normal hypergraphs.*

Corollary 1 becomes interesting in view of a recent result of Watrigant, Bougeret and Giroudeau [42], asserting that MVC is NP-hard when restricted to chordal graphs (one of the best-known classes of perfect graphs (see [8,27] and Section 4). Corollary 2 becomes interesting in view of the following remark. By inspecting Fig. 1, it follows that totally balanced hypergraphs and unimodular hypergraphs are incomparable proper subclasses of the class of balanced hypergraphs. The hypergraph $G = (\{0, 1, 2, 3\}, \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 1, 2, 3\}\})$ is totally balanced but not unimodular: $A(G)$ does not contain hole-submatrices but $\det A(G) = 2 \notin \{-1, 0, 1\}$. Polynomial-time algorithms for the Maximal Coverage problem on totally balanced hypergraphs are known for the special subclass of the so-called Rooted Path Tree hypergraphs [31], i.e, those hypergraphs (V, E) for which one can give an arborescence on V such that each $e \in E$ induces a directed path in this arborescence. However, the class of Rooted Path Tree hypergraphs, lies in the intersection between the class of totally balanced and unimodular hypergraphs [2] and therefore is properly contained in the class of unimodular hypergraphs. It follows that the following open problem is interesting.

Problem 1. Find the complexity of Maximal Coverage restricted to totally balanced hypergraphs.

More applications will be discussed in Section 4. The rest of the paper goes as follows. In the next section we give some notation and terminology. In Section 3 we prove our complexity result Theorem 1. The reduction strongly relies on a certain