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A branch-and-price algorithm for the robust graph coloring problem

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ABSTRACT

Given a graph G, an integer k, and a cost c_{uv} associated with all pairs uv of non-adjacent vertices in G, the robust graph coloring problem is to assign a color in $\{1, \ldots, k\}$ to every vertex of G so that no edge has both endpoints with the same color, and the total cost of the pairs of vertices having the same color is minimum. We propose a branch-and-price algorithm for the solution of this problem. The pricing problem consists in finding a stable set of minimum total weight, and we propose both an exact and a heuristic algorithm for its solution. Computational experiments are reported for randomly generated and benchmark graph coloring instances.

1. Introduction

The classical graph coloring problem is to assign a color to every vertex of a graph G so that no edge in G has both endpoints with the same color. Task assignment problems can be modeled as graph coloring problems, where each task is associated with a vertex, edges link vertices corresponding to overlapping tasks, and each color represents a person to be assigned to a set of tasks. The constraint imposing that no edge can have both endpoints with the same color is then equivalent to requiring that no two tasks overlapping in time are performed by the same person.

In practice, tasks are often subject to delays and it can therefore happen that two tasks assigned to the same person finally overlap, which makes the assignment infeasible. Task managers are therefore interested in determining a robust assignment that is not too much sensible to delays. For this purpose, one can for example associate a positive integer to each pair of non-overlapping tasks, to indicate the estimated probability that these two tasks will overlap in case of delays. Such a weighting procedure is illustrated in [8] for a crew assignment problem where crew teams have to be assigned to round-trip domestic flight routes from Hong Kong to other mainland China cities. Given two non-overlapping round trips T_i and T_j such that T_i precedes T_j, one can associate a weight $\omega_{ij} = \frac{g}{d-t+\alpha}$ to the pair $\{T_i, T_j\}$, where t is the return time of T_i, d is the departure time of T_j, g is the ground transfer time, and α is a constant. The problem is then to find a task assignment that minimizes the total weight of the pairs $\{T_i, T_j\}$ of round trips assigned to a same crew team.

Using graph theoretical terms, one can associate a positive weight to each pair of non-adjacent vertices, and then, rather than minimizing the number of colors needed to color the vertices of the graph, one can decide to use a fixed number of colors and try to minimize the total weight of the pairs of vertices having the same color. This is the Robust Graph Coloring Problem (RGCP) introduced by Ramírez-Rodríguez in his doctoral dissertation [12]. Applications of this extension of the classical

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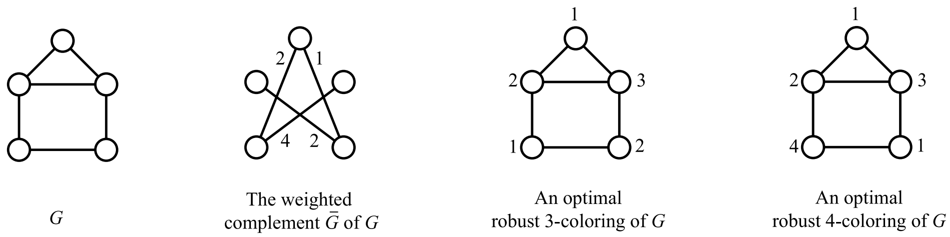

Fig. 1. Illustration of optimal robust colorings.

graph coloring problem are mentioned in [14], and range from the examination timetabling problem to cluster analysis. A mathematical formulation of the RGCP as a binary linear programming problem is proposed in [14,8] and the same authors also propose genetic algorithms to solve the problem. Another genetic algorithm, as well as a simulated annealing and a tabu search for the RGCP are described in [10] and a local search procedure is proposed in [6].

In this paper we propose a branch-and-price algorithm for the RGCP. A formal definition of the RGCP in given in the next section while the proposed exact algorithm is described in Section 3. Experimental results are reported in Section 4.

2. Problem definition

We are given an undirected graph $G = (V, E)$, where V is a finite set of vertices and E is a finite set of edges. Let $\bar{G}$ be the complement of G, i.e., $\bar{G} = (V, \bar{E})$, with $\bar{E} = \{uv : u, v \in V, u \neq v, uv \notin E\}$. A set S of vertices in G is *stable* if no edge in G has both endpoints in S, i.e. $uv \in \bar{E}$ for all $u, v \in S$. Given an integer k, a *k-coloring* of G is a partition of V into k stable sets, each stable set of the partition being called a *color class*. The *chromatic number* $\chi(G)$ of G is the smallest integer k such that there exists a k-coloring of G.

Assume that a non negative cost c_{uv} is associated to each pair $uv \in \bar{E}$. The Robust Graph Coloring Problem (RGCP) is the problem of finding a k-coloring of G for a given $k \geq \chi(G)$, such that the total cost of the pairs uv with u and v in a same color class is minimized. For illustration, a graph G, its complement $\bar{G}$, an optimal robust 3-coloring with total cost 4, and an optimal robust 4-coloring with total cost 1 are represented on Fig. 1.

3. A branch-and-price algorithm

In [14], Yáñez and Ramírez describe a binary linear programming model for the solution of the RGCP. They use the following binary variables:

- $x_{ui} = 1$ if color i is assigned to vertex u, 0 otherwise;
- $y_{uv} = 1$ if vertices u and v belong to the same color class, 0 otherwise.

The optimal solution of the RGCP can then be determined by solving the following problem:

$$\min \sum_{uv \in \bar{E}} c_{uv} y_{uv} \tag{1}$$

$$\sum_{i=1}^{k} x_{ui} = 1 \quad \forall u \in V \tag{2}$$

$$x_{ui} + x_{vi} \leq 1 \quad \forall uv \in E, \ \forall i \in \{1, \dots, k\} \tag{3}$$

$$x_{ui} + x_{vi} - 1 \leq y_{uv} \quad \forall uv \in \bar{E}, \ \forall i \in \{1, \dots, k\} \tag{4}$$

$$x_{ui} \in \{0, 1\} \quad \forall u \in V, \ \forall i \in \{1, \dots, k\} \tag{5}$$

$$y_{uv}\{0, 1\} \quad \forall uv \in \bar{E}. \tag{6}$$

Constraints (2) impose a color in $\{1, \dots, k\}$ on each vertex, constraints (3) ensure that each color class is a stable set and constraints (4) link the x variables with the y variables. As shown in [14], this model is not able to solve problems with more than 15 vertices.

We propose a set covering formulation for the RGCP. We denote by $\mathcal{S}$ the set of all stable sets in G, and for a stable set $s \in \mathcal{S}$, we define

- $c_s = \sum_{u,v \in s} c_{uv}$
- $\sigma_s =$ binary variable equal to 1 if s is chosen as a color class, 0 otherwise.