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Computing the differential of a graph: Hardness, approximability and exact algorithms

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ABSTRACT

We are studying computational complexity aspects of the differential of a graph, a graph parameter previously introduced to model ways of influencing a network. We obtain NP hardness results also for very special graph classes, such as split graphs and cubic graphs. This motivates to further classify this problem in terms of approximability. Here, one of our results shows MAXSNP completeness for the corresponding maximization problem on subcubic graphs. Moreover, we also provide a Measure & Conquer analysis for an exact moderately exponential-time algorithm that computes that graph parameter in time $O(1.755^n)$ on a graph of order n.

1. Introduction: definitions and simple properties

As explained in [3] in some detail, the differential of a graph can be seen as a simplified deterministic model of influencing a network representing a social group, aiming at maximizing the economic or political benefit of those who want to influence the network.

The central notions of the paper. We will use standard notions from graph theory throughout this paper. For instance, $N(x)$ denotes the set of vertices that are neighbors of the vertex x. Following [22], if D is a vertex set, then $B(D) = \left(\bigcup_{x\in D} N(x)\right) \setminus D$ is the *boundary* of D, collecting only the *proper neighbors* of vertices from D. The *differential of* D is defined as[1] $\partial(D) = |B(D)| - |D|$ and the *differential of a graph* Γ, written $\partial(\Gamma)$, is equal to $\max\{\partial(D) : D \subseteq V\}$. A set D satisfying $\partial(D) = \partial(\Gamma)$ is also called a *differential set*, or *∂-set* for short. If D has minimum cardinality among all ∂-sets, D is called a *minimum (cardinality) ∂-set*. The graph parameter ∂ was introduced in [22], where several basic properties were derived. We also refer to [3,4] and the literature quoted therein.

As explained in [22], the graph parameter $\partial(\Gamma)$ is related to the well-known parameter $\gamma(\Gamma)$ denoting the minimum size of a dominating vertex set in Γ, via a parameter Ψ known as the *enclaveless number* of a graph, namely [8,26],

$$\Psi(\Gamma) := \max\{|B(D)| : D \subseteq V\} = n - \gamma(\Gamma),$$

where n is the order of $\Gamma = (V, E)$, i.e., $n = |V|$. For a set D with $|B(D)| = \Psi(\Gamma)$, $B(D)$ is also known as a *nonblocker set* or as an *enclaveless set*.

Recall that a graph consisting of one central vertex c and d neighbors that in turn have no further neighbors other than c is also known as a *star* $S_d = K_{1,d}$. If $X = (V, E)$ is an S_d star with center c, then $V \setminus \{c\}$ will be also called *ray vertices* and the edges will be termed *rays*, in order to stay with the picture. We will call an S_d star *big* if $d \geq 2$.

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1 For bipartite graphs, this notion was already considered in [21] under the name of the *surplus* of a vertex set.

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Given a set $S \subseteq V$ and $x \in S$, we will say that $y \in V \setminus S$ is a *private neighbor* of x (with respect to S) if there is no other vertex $x' \in S$ such that y is also a neighbor of x'.

Simple observations. An alternative way of defining the differential of a graph is shown in the following proposition. Given a graph $\Gamma = (V, E)$, a *big star packing* is given by a vertex-disjoint collection $\mathcal{S} = \{X_i \mid 1 \le i \le k\}$ of (not necessarily induced) big stars $X_i \subseteq V$, i.e., $\Gamma[X_i]$ contains some S_d with $d = |X_i| - 1 \ge 2$. If $\mathcal{S}$ is a big star packing of Γ, we also denote this property by $\mathcal{S} \in SP(\Gamma)$.

Proposition 1.1 ([3]). $\partial(\Gamma) = \max\{\sum_{S\in\mathcal{S}}(|S| - 2) : \mathcal{S} \in SP(\Gamma)\}$.

The following structural observation is helpful in the following.

Lemma 1.2. (a) *Every vertex in a ∂-set D of Γ has at least one private neighbor with respect to D.*

(b) *Every vertex in a minimum ∂-set D of Γ has at least two private neighbors with respect to D.*

Proof. (a) Let D be a ∂-set. Assume that $x \in D$ has no private neighbor. Then, $D' = D \setminus \{x\}$ satisfies $B(D') = B(D)$. Hence, $\partial(D) < \partial(D')$, contradicting that $\partial(D) = \partial(\Gamma)$.

(b) Let D be a minimum ∂-set. Assume that $x \in D$ has only one private neighbor $y \notin D$. Then, $D' = D \setminus \{x\}$ obeys $B(D') = B(D) \setminus \{y\}$. Hence, $\partial(D) = \partial(D')$, but $|D'| < |D|$, contradicting the minimality of D. □

We collect some straightforward observations in the following remarks:

Remark 1.3. For a graph Γ of order n, $0 \le \partial(\Gamma) \le n - 2$. The lower bound is attained, e.g., for a collection of independent edges, and the upper bound is attained by the star S_{n-1}.

Remark 1.4. If Γ has connected components $\Gamma_1, \ldots, \Gamma_k$, then $\partial(\Gamma) = \partial(\Gamma_1) + \cdots + \partial(\Gamma_k)$.

It is often important to know the values of a graph parameter for special graphs. The next result gives the differential of paths and cycles.

Proposition 1.5 ([22]). *For paths P_n, $n \ge 1$ and cycles C_n, $n \ge 3$, $\partial(C_n) = \partial(P_n) = \left\lfloor \frac{n}{3} \right\rfloor$.*

Due to Remark 1.4 in combination with Proposition 1.5, it is easy to compute the differential of a graph with a maximum degree of two.

Proposition 1.6. *In a graph Γ of maximum degree (at most) two, $\partial(\Gamma)$ can be computed in polynomial time.*

The (easy) reasoning of the preceding proposition motivates why we will, often without saying, mostly consider connected graphs throughout this paper.

As we will show in this paper, the complexity picture changes if we ask to determine $\partial(\Gamma)$ for (sub)cubic or for split graphs Γ. This is interesting, as several graph parameters are computable in polynomial time on (sub)cubic or split graphs, although they are NP hard on general graphs, see [27,33].

Our main results. We derive in this paper two types of results: (1) We show computational hardness of the problem and several variants, especially restrictions to special graph classes. (2) We propose several algorithms for computing (approximations of) the differential of a graph. These results are placed into three different contexts, and this also provides the section structure of the paper. In Section 2, we study classical complexity questions and show that it is NP complete to decide, given a graph Γ and an integer k, whether $\partial(\Gamma) \ge k$ is true or not. This results also holds for split graphs and for cubic graphs. Section 3 is devoted to the study of (non-)approximability of the corresponding maximization problem. Among other results, we show MAXSNP completeness of this problem for (sub-)cubic graphs. We also obtain similar results for the problem of deciding whether $\Psi(\Gamma) \ge k$. In Section 4, we propose an exact algorithm for computing the differential of a graph. The running time has been analyzed by the Measure & Conquer paradigm, see [16]. As explained above, the graph parameter ∂ is related to domination, so that this exact algorithm nicely fits into the line of exact algorithms for domination-type problems; we only mention some of them in the following: [5,10,13,15,14,6,29,30,28,25].

2. Complexity of the differential of a graph

Given a graph $\Gamma = (V, E)$ and an integer k, we consider the following decision problem: Is $\partial(\Gamma) \ge k$? We refer to this problem as *k-Differential Set* (k-DS). A derived problem, called *Minimum Cardinality Maximum Differential Set* (MMDS), is to determine if there is a set D with $\partial(D) = \partial(\Gamma)$, $|D| \le l$ and $\partial(D) \ge k$, where k, l are two given parameters.[2]

We will prove that the two problems (k-DS MMDS) are NP complete in two well known (small) graph classes, split graphs and cubic graphs. Notice that the latter result is best possible in the sense that further restrictions on the degree would lead to graph classes where k-DS can be solved in polynomial time, as already mentioned in Proposition 1.6.

2 We prefer to keep this more descriptive naming over the possibly more consequent denotation (l, k)-DS.