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An upper bound on the sum of powers of the degrees of simple 1-planar graphs

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ABSTRACT

A 1-planar graph is a graph that can be drawn in the plane such that each edge is crossed by at most one other edge. For a fixed integer $k \geq 2$ and a simple 1-planar graph G on n vertices it is proven that $2(n-1)^k + O(n)$ is an upper bound on the sum of the k-th powers of the degrees of G.

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1. Introduction and result

We consider finite undirected and simple graphs G with vertex set $V(G)$ ($|V(G)| = n$), edge set $E(G)$ ($|E(G)| = m$), and refer to [9] for further notation and terminology. For a vertex $v \in V(G)$, let $N_G(v)$ be the set of neighbors of v in G and $d_G(v) = |N_G(v)|$ be the degree of v in G. Furthermore, for a nonnegative real number p let $\Sigma_p(G) = \sum_{v \in V(G)} d_G^p(v)$. Obviously, $\Sigma_0(G) = n$ and $\Sigma_1(G) = 2m$. The following results document that bounds on $\Sigma_p(G)$ are popular topics in research.

In [17], L.A. Székely et al. showed that $\Sigma_k(G) \leq (\Sigma_{\frac{1}{k}}(G))^k$, for every integer $k \geq 1$.

In [8], D. de Caen proved that $\Sigma_2(G) \leq m\left(\frac{2m}{n-1} + n - 2\right)$ for every simple graph G. Note that this bound is tight for complete graphs.

The investigation of eigenvalues of a graph is a prevalent topic in graph theoretical research. A survey of concerning results can be found in [3]. Bounds on $\Sigma_k(G)$ result in information on these eigenvalues. For example, let A, λ, $\overrightarrow{d}$, and δ be the adjacency matrix of G, the largest eigenvalue of A, the degree vector of G, and the minimum degree of G, respectively. Since A is a symmetric matrix, it is well known that the proposition

$$2m\delta^2 \leq 2 \sum_{uv \in E(G)} d_G(u) d_G(v) = \overrightarrow{d}^T A \overrightarrow{d} \leq \lambda \|\overrightarrow{d}\|^2 = \lambda \Sigma_2(G) \text{ holds.}$$

Together with de Caen's result, it leads to a lower bound on λ in terms of δ, n, and m.

De Caen's inequality is used by J.S. Li and Y.L. Pan in [14] to provide an upper bound on the largest eigenvalue of the Laplacian (the difference of the degree matrix and the adjacency matrix) of a graph.

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Sharp bounds for the moments $\frac{\Sigma_p(G)}{n}$ of the degree sequences of monotone families of graphs were obtained by Z. Füredi and A. Kündgen in [11].

De Caen's bound was generalized to hypergraphs by C. Bey in [2] and improved by K.Ch. Das in [7] to $\Sigma_2(G) \leq m\left(\frac{2m}{n-1} + \frac{n-2}{n-1}\Delta + (\Delta - \delta)(1 - \frac{\Delta}{n-1})\right)$, where Δ is the maximum degree of G.

In [4], S.M. Cioabă proved that $\Sigma_{p+1}(G) \leq \frac{2m}{n}\left(\Sigma_p(G) + (n-1)(\Delta^p - \delta^p)\right) - \frac{\Delta^p - \delta^p}{n}\Sigma_2(G)$ for any positive number p.

In the following, let $k \geq 2$ be an integer.

A simple planar graph on n vertices has at most $3n - 6$ edges. Hence, it follows from the bounds of de Caen and of Das (with $\Delta \leq n - 1$) that $\Sigma_2(G) \leq 3n^2 + O(n)$. In [5], R.J. Cook proved that $\Sigma_2(G) \leq 2n^2 + O(n)$ for a simple planar graph G on n vertices. Finally, in [12], for a simple planar graph G the upper bound

$$2(n-1)^k + 4^k(n-4) + 2 \cdot 3^k - 2((\delta + 1)^k - \delta^k)(3n - 6 - m) \quad \text{on } \Sigma_k(G) \text{ is established.}$$

In this paper we consider simple 1-planar graphs, i.e. simple graphs which can be drawn in the plane such that each edge is crossed by at most one other edge.

It follows from the result of Cioabă [4] that if $\varepsilon > 0$, G is simple and 1-planar, $k \geq 1$, and $n \geq (64 + 8\varepsilon)\varepsilon^{-1}$, then $\Sigma_k(G) < (8 + \varepsilon)n^k$.

The proof of this observation is by induction on k and based on a result in [1,10,15], that a simple 1-planar graph on n vertices has at most $4n - 8$ edges.

If $k = 1$, then $\Sigma_1(G) = 2m \leq 8n - 16 < (8 + \varepsilon)n$. Using $m \leq 4n - 8$, $\Delta \leq n - 1$, and Cioabă's result, $\Sigma_{k+1}(G) \leq \frac{8n-16}{n}(\Sigma_k(G) + (n-1)^{k+1}) < 8((8 + \varepsilon)n^k + n^{k+1}) \leq (8 + \varepsilon)n^{k+1}$.

Our main result Theorem 1.1 shows that this result can be improved if n is large enough.

If G is a simple planar graph, then, using $m \leq 3n - 6$, it follows from the mentioned result in [12] that $\Sigma_k(G) \leq 2(n-1)^k + O(n)$ for fixed k. Moreover, in [12], simple planar graphs are constructed showing that this bound is asymptotically tight. Our result Theorem 1.1 implies that this result remains true if the class of simple planar graphs is extended to simple 1-planar graphs.

Theorem 1.1. *If $k \geq 2$ is a fixed integer and G is a simple 1-planar graph on n vertices, then $\Sigma_k(G) \leq 2(n-1)^k + O(n)$. Moreover, this bound is asymptotically tight.*

The rest of the paper is organized as follows. Notation and terminology given in Section 1 remain valid for the next sections. Section 2 starts with definitions of drawings, crossings, and 1-planar graphs. Then, in preparation for the proof of the main result, namely Theorem 1.1, some lemmas are presented. Finally, the proof of Theorem 1.1 is postponed to Section 3. In Section 4 we conclude with some remarks on q-planar graphs for q greater than 1.

2. Preliminaries

In [16], J. Pach and G. Tóth present results on drawings of graphs in the plane and related crossing numbers. The following definitions, being sufficient for our purposes here, are taken from there. For more details see e.g. [13,15,16].

A *drawing* of a simple undirected graph is a mapping f that assigns to each vertex a distinct point in the plane and to each edge uv a continuous arc (i.e., a homeomorphic image of a closed interval) connecting $f(u)$ and $f(v)$, not passing through the image of any other vertex. For simplicity, the arc assigned to uv is called an *edge* of the drawing, and if it leads to no confusion, it is also denoted by uv. We assume that no three edges have an interior point in common, and if two edges share an interior point P, then they *properly cross* at P. We also assume that any two edges of a drawing have only a finite number of *crossings* (common interior points). A common endpoint of two edges does not count as a crossing.

It is easy to see that we may assume that no two adjacent edges cross each other and that two nonadjacent edges cross each other at most once, since such crossings can be gotten rid of easily without introducing new crossing pairs in case of 1-planar drawings.

A drawing of a graph G in the plane such that each edge is crossed by at most one other edge is called a 1-*planar drawing* of G and G itself is called 1-*planar* in this case. The class of 1-planar graphs is well investigated in the literature. For properties of these graphs see e.g. [1,10,15,16]. We will use the obvious property that any subgraph of a 1-planar graph is also 1-planar.

The 1-planarity of complete multipartite graphs is investigated in [6]; Lemma 2.1 is taken from there.

Lemma 2.1 ([6])**.** *The graphs $K_{3,7}$ and $K_{4,5}$ are not 1-planar.*

In the following, the following Lemma 2.2 will be frequently used.

Lemma 2.2 ([1,10,15])**.** *Any simple 1-planar graph G has at most $4|V(G)| - 8$ edges, and hence, minimum degree at most 7.*

Lemma 2.3. *Let G be a simple 1-planar graph on n vertices.*

Then $\sum_{x \in N_G(v)} d_G(x) \leq 2n + 605$ for any vertex $v \in V(G)$ with $d_G(v) \leq 7$.

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