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Feasibility Pump-like heuristics for mixed integer problems

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ABSTRACT

Finding a feasible solution to a MIP problem is a tough task that has received much attention in the last decades. The Feasibility Pump (FP) is a heuristic for finding feasible solutions to MIP problems that has encountered a lot of success as it is very efficient also when dealing with very difficult instances. In this work, we show that the FP heuristic for general MIP problems can be seen as the Frank–Wolfe method applied to a concave nonsmooth problem. Starting from this equivalence, we propose concave non-differentiable penalty functions for measuring solution integrality that can be integrated in the FP approach.

1. Introduction

Finding a first feasible solution quickly is crucial for solving Mixed Integer Programming (MIP) problems as many local-search approaches can be used only if a feasible solution is available (see e.g. [11,16]).

Several heuristic methods for finding a first feasible solution for a MIP problem have been proposed in the literature (see e.g. [4,14,18–23]). In particular, the Feasibility Pump [15] is considered one of the most efficient heuristics available.

The Feasibility Pump approach generates two sequences of points $\{\bar{x}^k\}$ and $\{\tilde{x}^k\}$ such that $\bar{x}^k$ is LP-feasible, but may not be integer feasible, and $\tilde{x}^k$ is integer, but not necessarily LP-feasible. To be more specific the algorithm starts with a solution of the LP relaxation $\bar{x}^0$ and sets $\tilde{x}^0$ equal to the rounding of $\bar{x}^0$. Then, at each iteration $\bar{x}^{k+1}$ is chosen as the nearest LP-feasible point in the ℓ_1-norm to $\tilde{x}^k$, and $\tilde{x}^{k+1}$ is obtained as the rounding of $\bar{x}^{k+1}$. The idea of the algorithm is to reduce at each iteration the distance between the points of the two sequences, until the two points are the same and an integer feasible solution is found. Unfortunately, it can happen that the distance between $\bar{x}^{k+1}$ and $\tilde{x}^k$ is greater than zero and $\tilde{x}^{k+1} = \tilde{x}^k$, and the strategy can stall. In order to overcome this drawback, random perturbations and restart procedures are performed.

Various papers devoted to further improvements of the Feasibility Pump have been developed. In [5], the authors extended the Feasibility Pump in order to handle MIP problems with both 0–1 and integer variables and they further exploited the FP information to drive a subsequent enumeration phase. Achterberg and Berthold [1] proposed a different distance function which takes into account the original objective function in order to improve the quality of the feasible solution found. Some efforts have also been made to propose new rounding techniques [3,6,17]. In [3,6] the main idea is that of choosing a rounded solution along a line segment passing through the LP-feasible solution. In [17], an exact method combining a diving-like procedure and constraint propagation is proposed.

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For the case of 0–1 MIPs an interesting interpretation of the FP has been given by Eckstein and Nediak in [14]: they noticed that the FP heuristic may be seen as a form of Frank–Wolfe procedure applied to a nonsmooth merit function. This concave nonsmooth function penalizes the violation of the integrality constraints but its reduction might not correspond to a decrease in the number of variables that violate integrality. Starting from this interpretation in [12] the relationship between the Feasibility Pump and the Frank–Wolfe algorithm has been exploited to define a new version of the Feasibility Pump obtained by applying the Frank–Wolfe algorithm to a different nonsmooth concave function which penalizes the violation of the integrality constraints and has the good property that its reduction cannot correspond to an increase in the number of fractional variables.

In this work, we first show that the equivalence between the FP and the Frank–Wolfe algorithm still holds for the general integer case. Then we extend the results proposed in [12] for 0–1 MIPs to general mixed integer problems. The extension is not straightforward. In the 0–1 case, due to the fact that the integer feasible points lie on the boundary of the relaxed feasible set, we can use a suitable class of concave penalty functions and a Frank–Wolfe based approach to find an integer feasible solution. The first choice depends on the fact that, when minimizing a concave function over a polyhedron, the global optima, if any, are on the boundary of the polyhedron. On the other hand, the choice of the Frank–Wolfe approach is motivated by the fact that it is well suited for the class of problems we want to solve, since in this case the algorithm moves from a vertex to another until it finds a stationary point. When dealing with general MIP problems, there are integer feasible points that are in the interior of the feasible set. So, the direct extension of the approach considered in [12] would lead to penalty functions that are not concave anymore (the functions admit global minima in the interior of the feasible set). This would further imply a loss in the efficiency of the Frank–Wolfe algorithm, as we should include a suitable line search technique to guarantee the convergence of the method. In order to overcome these issues, we need to exploit the hidden concavity of the penalty problem obtained with the approach described in [12] by considering a new equivalent concave problem having a larger number of variables.

The paper is organized as follows. In Section 2, we give a brief review of the Feasibility Pump heuristic for general MIP problems. In Section 3, we show the equivalence between the FP heuristic and the Frank–Wolfe algorithm applied to a nonsmooth merit function. In Section 4, we introduce new nonsmooth merit functions for dealing with general integer variables, and discuss their properties. We present our algorithm in Section 5, and in Section 6 we explain how it can be integrated in the Objective Feasibility Pump [1]. In Section 7 we give a performance comparison of our algorithm with the Objective FP showing that the merit functions proposed can improve the efficiency of the FP approach in terms of CPU time.

In the following, given a concave function $f : R^n \to R$, we denote by $\partial f(x)$ the set of supergradients of f at the point x, namely

$$\partial f(x) = \{v \in R^n : f(y) - f(x) \le v^T(y - x), \forall y \in R^n\}.$$

2. The Feasibility Pump heuristic for general MIP problems

We consider a MIP problem of the form:

$$\begin{aligned} &\min\ c^T x \\ &\text{s.t.}\ Ax \ge b \\ &l \le x \le u \\ &x_j \in \mathbb{Z}, \quad \forall j \in I, \end{aligned} \tag{MIP}$$

where $A \in \mathbf{R}^{m\times n}$ and $I \subseteq \{1, 2, \ldots, n\}$ is the set of indices of the integer variables. Let $P = \{x : Ax \ge b, l \le x \le u\}$ denote the polyhedron of the LP-relaxation of (MIP). The Feasibility Pump [5,15] starts from the solution of the LP relaxation problem $\bar{x}^0 := \arg\min\{c^T x : x \in P\}$ and generates two sequences of points $\bar{x}^k$ and $\tilde{x}^k$: $\bar{x}^k$ is LP-feasible, but may be integer infeasible; $\tilde{x}^k$ is integer, but not necessarily LP-feasible. At each iteration $\bar{x}^{k+1} \in P$ is the nearest point in ℓ_1-norm to $\tilde{x}^k$:

$$\bar{x}^{k+1} := \arg\min_{x\in P}\ \Delta(x, \tilde{x}^k) \tag{1}$$

where

$$\Delta(x, \tilde{x}^k) = \sum_{j\in I} |x_j - \tilde{x}^k_j|.$$

The point $\tilde{x}^{k+1}$ is obtained as the rounding of $\bar{x}^{k+1}$. The procedure stops if at some iteration h, $\bar{x}^h$ is integer or, in case of failing, if it reaches a time or iteration limit. In order to avoid stalling issues and loops, the Feasibility Pump performs a perturbation step. Here we report a brief outline of the basic scheme [5,15]: