Discrete Applied Mathematics 165 (2014) 168–174

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Co-TT graphs and a characterization of split co-TT graphs

Martin Charles Golumbic [a,*], Nirit Lefel Weingarten [a], Vincent Limouzy [b]

[a] Caesarea Rothschild Institute and Department of Computer Science, University of Haifa, Israel

[b] University of Blaise Pascal - LIMOS, France

ARTICLE INFO

Article history:
Received 1 November 2011
Received in revised form 4 November 2012
Accepted 23 November 2012
Available online 11 December 2012

Dedicated to the memory of Bruno Simeone

Keywords:
Complement threshold tolerance graph
Interval containment bigraph
Interval graph
Split graph

ABSTRACT

In this paper, we present a new characterization of complement Threshold Tolerance graphs (co-TT for short) and find a recognition algorithm for the subclass of split co-TT graphs running in $O(n^2)$ time. Currently, the best recognition algorithms for co-TT graphs and for split co-TT graphs run in $O(n^4)$ time (Hammer and Simeone (1981) [4]; Monma et al. (1988) [7]).

1. Introduction

In this paper, we introduce new characterizations of complement Threshold Tolerance graphs (co-TT graphs) and of the subclass of split co-TT graphs (split graphs $\bigcap$ co-TT graphs). In the latter case, the characterization provides a recognition algorithm for split co-TT graphs that runs in $O(n^2)$ time. Currently, the best recognition algorithms for co-TT graphs and split co-TT graphs run in $O(n^4)$ time due to [4,7]. We start by defining graphs which have a blue–red partition and immediately conclude that these graphs are exactly co-TT graphs. We characterize a co-TT graph by representing its vertices by a family of blue and red intervals, such that,

- the blue intervals form an intersection model of an interval induced subgraph,
- the red intervals form a stable set in the graph, and
- the blue and the red intervals together form a containment model of an interval containment bigraph which is an induced subgraph.

Then, we describe how a blue–red partition of the vertices of a co-TT graph can be found efficiently. We characterize a split co-TT graph by representing its vertices with a family of blue and red intervals, such that,

- the blue intervals form an intersection model of a clique in the graph,
- the red intervals form a stable set in the graph, and
- the blue and the red intervals together form a containment model of an interval containment bigraph which is an induced subgraph.

Finally, we describe the recognition algorithm for split co-TT graphs using our characterizing blue–red partition and a theorem of Jing Huang [5].

* Corresponding author. Tel.: +972 4 8257792; fax: +972 4 8288181.
E-mail addresses: golumbic@cs.haifa.ac.il (M.C. Golumbic), niritlefel@gmail.com (N.L. Weingarten), limouzy@isima.fr (V. Limouzy).

0166-218X/$ – see front matter
doi:10.1016/j.dam.2012.11.014

2. Preliminaries and notations

For the sake of convenience, we use the standard graph theoretical definitions from [2,3]. We consider finite undirected graphs $G = (V, E)$ with no loops nor multiple edges.

The *(open) neighborhood* $N(x)$ of a vertex x in $G = (V, E)$ is the set of vertices adjacent to x, and the *closed neighborhood* $N[x]$ of x is the union of x together with its open neighborhood. Two vertices x, y of a graph are called *true twins* when they have the same closed neighborhood, i.e., $N[x] = N[y]$. Two vertices x, y of a graph are called *false twins* when they have the same open neighborhood, i.e., $N(x) = N(y)$. For a subset U of the vertices, the *induced* subgraph $G_U = (U, E_U)$ has $E_U = \{xy \in E \mid x, y \in U\}$.

A *clique* in the graph $G = (V, E)$ is a set of vertices such that every two vertices in the set are connected by an edge. A *stable set* in the graph $G = (V, E)$ is a set of vertices such that every two vertices in the set are not connected by an edge. A graph is a *split* graph if its vertices can be partitioned into a stable set and a clique. A graph is a *bipartite graph* if its vertices can be partitioned into two stable sets. We often denote a bipartite graph whose partition has the parts X and Y as $H = (X, Y, E)$. A vertex x is called a *simplicial vertex* if its neighborhood $N(x)$ is a clique. A graph $G = (V, E)$ is an *interval* graph if there is a family of intervals $\{I_v \mid v \in V\}$, such that for $x, y \in V$, $xy \in E$ if and only if the intersection between I_x and I_y is nonempty. Similarly, G is a *circular arc* graph if it is the intersection graph of arcs on a circle.

An undirected graph $G = (V, E)$ is called a *Threshold Tolerance (TT) graph* if its vertices can be assigned positive weights $\{w_v \mid v \in V\}$ and positive tolerances $\{t_v \mid v \in V\}$ such that $xy \in E \Leftrightarrow w_x + w_y \geq \min\{t_x, t_y\}$. The *complement of a Threshold Tolerance graph* is called a *co-TT graph*. An alternate definition of co-TT graphs is:

Definition 2.1 (*Co-TT Graphs—Monma, Reed and Trotter [7]*). A graph $G = (V, E)$ is a *complement Threshold Tolerance graph (co-TT)* if for each vertex v there exist two positive numbers a_v and b_v such that for any pair of vertices x and y:

$$xy \in E \iff a_x \leq b_y \quad \text{and} \quad a_y \leq b_x.$$

(See also [3, p. 187].)

Definition 2.2 (*Split Co-TT Graphs*). A graph G is a *split co-TT* graph if it is both a co-TT graph and a split graph, i.e., $G \in$ *split* $\bigcap$ *co-TT*.

Definition 2.3 (*Interval Containment Bigraph—Huang [5]*). A bipartite graph $H = (X, Y, E)$ is an *interval containment bigraph* if there is a family of intervals $\{I_v \mid v \in X \cup Y\}$, such that for all $x \in X$ and $y \in Y$, $xy \in E$ if and only if I_x contains I_y.

The following theorem by Jing Huang [5] characterizes interval containment bigraphs; it is central to the proof of correctness of our split co-TT recognition algorithm.

Theorem 2.4 (*Huang [5]*). *Let $H = (X, Y, E)$ be a bipartite graph. Then the following statements are equivalent:*

1. *H is an interval containment bigraph with respect to (X, Y),*
2. *There is a family of intervals $\{I_v \mid v \in X \cup Y\}$, such that*
   - *all intervals I_v intersect,*
   - *for any $x \in X$ and $y \in Y$, x and y are adjacent if and only if I_x contains I_y.*
3. *The complement of H is a circular arc graph.*

3. Co-TT characterization

We start by giving an alternative definition for an interval graph:

Definition 3.1 (*Interval Graph*). A graph is an *interval graph* if for each vertex v there exist two positive numbers a_v and b_v such that $a_v \leq b_v$ and for any pair of vertices x and y:

$$xy \in E \iff a_x \leq b_y \quad \text{and} \quad a_y \leq b_x.$$

For example, Fig. 1 shows an interval graph and its intervals representation. There is an edge between w and t because $w \mapsto (a_w = 10, b_w = 12)$, $t \mapsto (a_t = 7, b_t = 11)$ and $a_w \leq b_t \wedge a_t \leq b_w$. There is no edge between z and w because $z \mapsto (a_z = 3, b_z = 9)$ and $a_w > b_z$.

Definition 3.2 (*Blue–Red Partition of a Graph*). A partition $V = B \bigcup R$ of the vertices of a graph $G = (V, E)$ is a *blue–red partition* if for each vertex v of G there exist two positive numbers a_v and b_v, inducing a partition of the vertex set into two disjoint sets as follows: the blue vertices B are the vertices v such that $a_v \leq b_v$, the red vertices R are the vertices v such that $b_v < a_v$ and