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A reduction algorithm for the weighted stable set problem in claw-free graphs

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ABSTRACT

In this article the Lovász–Plummer *clique reduction* is extended to the weighted case and used to find a maximum weight stable set in a claw-free graph G with n nodes in $\mathcal{O}(n^2(n^2 + \mathcal{L}(n)))$ time, where $\mathcal{L}(n)$ is the complexity of finding a maximum weight augmenting path in a line graph H with n nodes. The best algorithm known to date to solve the latter problem is Gabow's maximum weight matching algorithm (applied to the root graph of H) which has a complexity of $\mathcal{O}(n^2 \log n)$. It follows that our algorithm can produce a maximum weight stable set in a claw-free graph in $\mathcal{O}(n^4 \log n)$ time.

1. Introduction

A *Matching* in a graph $G(V, E)$ is a family of pairwise node-disjoint *edges*. If G is an *edge-weighted* graph, the *Maximum Weight Matching Problem (MWMP)* asks for a matching with maximum total weight. Since Edmonds seminal paper [3] featuring the *blossom's shrinking* technique, many polynomial algorithms of increasing ingenuity and decreasing complexity have been proposed in the literature, both for the weighted and the unweighted case [14].

The *Maximum Weight Stable Set Problem (MWSSP)* in a graph $G(V, E)$ with node-weight function $w : V \rightarrow \Re$ asks for a maximum weight subset of pairwise non-adjacent *nodes*. The Maximum Weight Matching Problem is a special case of the Maximum Weight Stable Set Problem. In fact, the latter can be transformed into the former in a very specific class of graphs: the *line graphs*. The line graph of a graph $G(V, E)$ is the graph $L(G)$ with node set E and an edge ef for each pair $\{e, f\}$ of edges of E incident to the same node. A graph G is a *line graph* if and only if there exists a graph H (the *root graph* of G) with the property that $L(H) = G$.

Evidently, M is a matching in G if and only if the nodes corresponding to the edges of M define a stable set in $L(G)$. This implies that the crucial optimality condition for the weighted matching problem based on the concept of *alternating path (cycle)* can be extended to line graphs and that the Maximum Weight Stable Set Problem, which is NP-Hard on general graphs, can be solved in polynomial time if G is a line graph.

Indeed one can do more [1], and extend the *augmenting path condition* to a class of graphs that properly contains line graphs: the *claw-free graphs*. A graph $G(V, E)$ is *claw-free* if no vertex $v \in V$ has three mutually non-adjacent nodes in its neighborhood. Berge observed that the optimality condition that holds for matchings in any graph can be extended to stable sets in claw-free graphs. In particular, by letting $w(T) = \sum_{v \in T} w(v)$ (where $T \subseteq V$) we have that a stable set S in a claw-free graph has maximum weight if and only if there does not exist an alternating path (cycle) P in G with $w(P \setminus S) - w(P \cap S) > 0$.

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In 1980, building upon the augmenting path condition, Minty [11] proposed a $\mathcal{O}(|V|^6)$ algorithm to find a maximum weight stable set in a claw-free graph. Minty's crucial idea was that of defining a new map (different from the line transformation $L(G)$) from the node-weighted graph G to an edge-weighted graph H with the property that a stable set has maximum (node) weight in G if and only if a suitable matching has maximum (edge) weight in H. For more than twenty years and despite the great interest surrounding the stable set problem in claw-free graphs, no algorithm better than Minty's was proposed. In 1993 Pulleyblank–Shepherd [13] proposed a $\mathcal{O}(|V|^4)$ algorithm for the MWSSP in *distance claw-free* graphs. In 2001 Nakamura and Tamura observed that Minty's algorithm had a minor flaw in its construction and proposed a simple way to fix it [12]. Subsequently, Schrijver [14], elaborating on Minty's approach, proposed an elegant alternative using a slightly different edge-weighted auxiliary graph H. A fresh interest in the algorithmic issues concerning claw-free graphs was brought forward by the deep structural description of this class of graphs proposed by Chudnovsky and Seymour [2]. Inspired by those results, Faenza, Oriolo, Pietropaoli and Stauffer [4], using an entirely new approach, have recently proposed a $\mathcal{O}(|V|^4)$ decomposition algorithm and, more recently, an astonishing $\mathcal{O}(|V|^3)$ update to the latter result has been proposed by Faenza, Oriolo and Stauffer [5].

This paper tries to combine the basic ideas of Minty's algorithm with a fundamental structural result proved by Lovász and Plummer [10]: the *clique reduction*. More specifically, after showing, in the spirit of Minty, that the quest for a maximum weight augmenting path in a claw-free graph can be restricted to a subgraph of G (the *augmenting subgraph*), we prove that the augmenting subgraph can be, in turn, *reduced to a line graph* by repeatedly applying Lovász–Plummer reduction (appropriately extended to cope with the weighted case). As a consequence we are able to find a maximum weight augmenting path in G in $\mathcal{O}(n^2+\mathcal{L}(n))$ time, where $\mathcal{L}(n)$ is the complexity of finding a maximum weight augmenting path in a line graph with n nodes.

The main result is eventually achieved by proving that $\mathcal{O}(n^2)$ maximum weight augmenting paths have to be computed in order to find a maximum weight stable set in a claw-free graph G.

For each graph $G(V,E)$ we denote by $E_G(U)$ the edges of G with endnodes in $U \subseteq V$, by $G[U]$ the subgraph of G induced by U and by $N_G(U)$ the set of nodes in $V \setminus U$ adjacent in G to some node in U. If $U = \{u\}$ we simply write $N_G(u)$. We denote by $N_G[U]$ and $N_G[u]$ the sets $N_G(U) \cup U$ and $N_G(u) \cup \{u\}$. Moreover, we denote by $N^2_G(U)$ and $N^2_G(u)$ the sets $N_G(N_G(U)) \setminus U$ and $N_G(N_G(u)) \setminus \{u\}$. When no confusion arises we omit the subscripts and write $E(U)$, $N(U)$, $N(u)$, $N[U]$, $N[u]$, $N^2(U)$, $N^2(u)$. To simplify the notation, we denote an edge $(u,v) \in E$ also as uv. A *clique* is a set of nodes which induces a complete subgraph of G. The *symmetric difference* $(U \setminus V) \cup (V \setminus U)$ of two sets U, V is denoted by $U \Delta V$. To highlight its structure, a *claw* induced by a node u adjacent to three nodes x, y, z mutually non adjacent is denoted as $(u : x, y, z)$. A P_k is a (chordless) path induced by k nodes and will be denoted as $(u_1, \ldots, u_k)$. A node $u \in V$ is said to be *complete* (respectively *anti-complete*) to a subset $U \subseteq V$ if $U \subseteq N(u)$ (respectively $U \cap N(u) = \emptyset$).

A graph is a *line graph* if and only if there exists a family of cliques (*Krausz partition*) covering all of its edges and with the property that every node belongs to *exactly* two cliques of the family (Krausz [8]; see also [10], exercise 12.4.2). A graph is *quasi-line* if the neighborhood of each node can be covered by two cliques. Each line-graph is a quasi-line graph and each quasi-line graph is a claw-free graph. A 5-wheel $W_5 = (\bar{v}; v_0, v_1, v_2, v_3, v_4)$ is a graph consisting of a 5-hole $R = \{v_0, v_1, v_2, v_3, v_4\}$ called *rim* of W_5 and the node $\bar{v}$ (*hub* of W_5) adjacent to every node of R.

In the rest of this paper we tacitly assume that every graph $G(V,E)$ has $|V| = n$, $|E| = m$ and is claw-free. Moreover, in what follows we will consider special subgraphs of G (paths, cycles, wheels, cliques, etc.) both as graphs and subsets of V. Hence, for example, if P is a path in G and v a node in P, we also use the shorthand notation $v \in P \subseteq V$. If S is a stable set of $G(V,E)$ then any node $v \in V \setminus S$ satisfies $|N(v) \cap S| \le 2$ and is called *superfree* if $|N(v) \cap S| = 0$, *free* if $|N(v) \cap S| = 1$ and *bound* if $|N(v) \cap S| = 2$.

Following Minty, we call *wing* of $\{s,t\} \subseteq S$, S stable set, the set $W(s,t) = \{u \in V \setminus S : N(u) \cap S = \{s,t\}\}$. The nodes s and t are said to be the *extrema* of the wing. By claw-freeness $\alpha(W(s,t)) \le 2$; if $\alpha(W(s,t)) = 1$ the wing $W(s,t)$ is said to be a *clique-wing*. An *x–y-alternating path* with respect to S is an induced path $P = \{x, \ldots, y\}$ whose nodes alternate between S and $V \setminus S$ and x (y) is free or belongs to S. Evidently, if P is an alternating path, then the set $S' = (S \setminus P) \cup (P \setminus S)$ is a stable set in G. An x–y-alternating path P such that x and y are both free is an *x–y-augmenting path*. The value $\Delta_w(P) = w(P \setminus S) - w(P \cap S)$ is said to be the *weight* of P.

2. Reducible and strongly reducible cliques

In this section we shall assume that S is a maximal stable set of a claw-free graph $G(V,E)$. A maximal clique Q is *reducible* if $\alpha(N(Q)) \le 2$. Two non-adjacent nodes $u, v \in N(Q)$ are *distant* (with respect to Q) if $N(u) \cap N(v) \cap Q = \emptyset$. A (maximal) clique is *normal* if it has three neighbors that are mutually distant. In [10] Lovász and Plummer defined the *reduction* of a reducible clique and proved that it preserves claw-freeness. Moreover, they proved the following crucial results:

Theorem 1. *If $G(V,E)$ is a claw-free graph that does not contain an induced 5-wheel, one has:*

(i) *two independent neighbors of a* normal *clique of G are* distant*;*
(ii) *if a node $v \in V$ is contained in two different* irreducible *cliques Q_1 and Q_2 and $Q_1 \cup Q_2 \supseteq N(v)$ then Q_1 and Q_2 are* normal*;*
(iii) *if $v \in V$ is contained in two different* normal *cliques Q_1 and Q_2 then $N(v) \subseteq Q_1 \cup Q_2$;*
(iv) *if each node $v \in V$ is contained in two different* irreducible *cliques Q_1 and Q_2 and $N(v) = Q_1 \cup Q_2$, then G is a* line graph. □