Discrete Applied Mathematics 165 (2014) 263–269

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On sum edge-coloring of regular, bipartite and split graphs

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ARTICLE INFO

Article history:
Received 30 November 2011
Received in revised form 16 August 2013
Accepted 27 September 2013
Available online 21 October 2013

Keywords:
Edge-coloring
Sum edge-coloring
Regular graph
Bipartite graph
Split graph

ABSTRACT

An edge-coloring of a graph G with natural numbers is called a sum edge-coloring if the colors of edges incident to any vertex of G are distinct and the sum of the colors of the edges of G is minimum. The edge-chromatic sum of a graph G is the sum of the colors of edges in a sum edge-coloring of G. It is known that the problem of finding the edge-chromatic sum of an r-regular ($r \geq 3$) graph is NP-complete. In this paper we give a polynomial time $\left(1+\frac{2r}{(r+1)^2}\right)$-approximation algorithm for the edge-chromatic sum problem on r-regular graphs for $r \geq 3$. Also, it is known that the problem of finding the edge-chromatic sum of bipartite graphs with maximum degree 3 is NP-complete. We show that the problem remains NP-complete even for some restricted class of bipartite graphs with maximum degree 3. Finally, we give upper bounds for the edge-chromatic sum of some split graphs.

1. Introduction

We consider finite undirected graphs that do not contain loops or multiple edges. Let $V(G)$ and $E(G)$ denote sets of vertices and edges of G, respectively. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S, that is, $V(G[S]) = S$ and $E(G[S])$ consists of those edges of $E(G)$ for which both ends are in S. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of G by $\Delta(G)$, the chromatic number of G by $\chi(G)$, and the chromatic index of G by $\chi'(G)$. The terms and concepts that we do not define can be found in [2,26].

A proper vertex-coloring of a graph G is a mapping $\alpha : V(G) \to \mathbf{N}$ such that $\alpha(u) \neq \alpha(v)$ for every $uv \in E(G)$. If α is a proper vertex-coloring of a graph G, then $\Sigma(G,\alpha)$ denotes the sum of the colors of the vertices of G. For a graph G, define the vertex-chromatic sum $\Sigma(G)$ as follows: $\Sigma(G) = \min_\alpha \Sigma(G,\alpha)$, where minimum is taken among all possible proper vertex-colorings of G. If α is a proper vertex-coloring of a graph G and $\Sigma(G) = \Sigma(G,\alpha)$, then α is called a sum vertex-coloring. The strength of a graph G ($s(G)$) is the minimum number of colors needed for a sum vertex-coloring of G. The concept of sum vertex-coloring and vertex-chromatic sum was introduced by Kubicka [16] and Supowit [22]. In [18], Kubicka and Schwenk showed that the problem of finding the vertex-chromatic sum is NP-complete in general and polynomial time solvable for trees. Jansen [12] gave a dynamic programming algorithm for partial k-trees. In papers [5,6,9,13,17], some approximation algorithms were given for various classes of graphs. For the strength of graphs, Brook's-type theorem was proved in [11]. On the other hand, there are graphs with $s(G) > \chi(G)$ [8]. Some bounds for the vertex-chromatic sum of a graph were given in [23].

Similar to the sum vertex-coloring and vertex-chromatic sum of graphs, in [5,10,11], sum edge-coloring and edge-chromatic sum of graphs were introduced. A proper edge-coloring of a graph G is a mapping $\alpha : E(G) \to \mathbf{N}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e, e' \in E(G)$. If α is a proper edge-coloring of a graph G, then $\Sigma'(G,\alpha)$ denotes

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http://dx.doi.org/10.1016/j.dam.2013.09.025

the sum of the colors of the edges of G. For a graph G, define the edge-chromatic sum $\Sigma'(G)$ as follows: $\Sigma'(G) = \min_{\alpha} \Sigma'(G, \alpha)$, where minimum is taken among all possible proper edge-colorings of G. If α is a proper edge-coloring of a graph G and $\Sigma'(G) = \Sigma'(G, \alpha)$, then α is called a sum edge-coloring. The edge-strength of a graph G ($s'(G)$) is the minimum number of colors needed for a sum edge-coloring of G. For the edge-strength of graphs, Vizing's-type theorem was proved in [11]. In [5], Bar-Noy et al. proved that the problem of finding the edge-chromatic sum is NP-hard for multigraphs. Later, in [10], it was shown that the problem is NP-complete for bipartite graphs with maximum degree 3. Also, in [10], the authors proved that the problem can be solved in polynomial time for trees and that $s'(G) = \chi'(G)$ for bipartite graphs. In [20], Salavatipour proved that the problem of determining the edge-chromatic sum and the problem of determining the edge-strength are both NP-complete for r-regular graphs with $r \geq 3$. Also he proved that $s'(G) = \chi'(G)$ for regular graphs. On the other hand, there are graphs with $\chi'(G) = \Delta(G)$ and $s'(G) = \Delta(G) + 1$ [11]. Recently, Cardinal et al. [7] determined the edge-strength of the multicycles.

In the present paper we give a polynomial time $\frac{11}{8}$-approximation algorithm for the edge-chromatic sum problem of r-regular graphs for $r \geq 3$. Next, we show that the problem of finding the edge-chromatic sum remains NP-complete even for some restricted class of bipartite graphs with maximum degree 3. Finally, we give upper bounds for the edge-chromatic sum of some split graphs.

2. Definitions and preliminary results

A proper t-coloring is a proper edge-coloring which makes use of t different colors. If α is a proper t-coloring of G and $v \in V(G)$, then $S(v, \alpha)$ denotes the set of colors appearing on edges incident to v. Let G be a graph and $R \subseteq V(G)$. A proper t-coloring of a graph G is called an R-sequential t-coloring [1,3] if the edges incident to each vertex $v \in R$ are colored by the colors $1, \ldots, d_G(v)$. For positive integers a and b, we denote by $[a, b]$, the set of all positive integers c with $a \leq c \leq b$. For a positive integer n, let K_n denote the complete graph on n vertices.

We will use the following four results.

Theorem 1 ([15]). *If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.*

Theorem 2 ([24]). *For every graph G,*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Theorem 3 ([25]). *For the complete graph K_n with $n \geq 2$,*

$$\chi'(K_n) = \begin{cases} n-1, & \text{if } n \text{ is even}, \\ n, & \text{if } n \text{ is odd}. \end{cases}$$

Theorem 4 ([10,11]). *If G is a bipartite or a regular graph, then $s'(G) = \chi'(G)$.*

We also need one result on the edge-chromatic sum of complete graphs with shifted colors. First we give a definition of the shifted edge-chromatic sum. If α is a proper t-coloring of a graph G with colors $[p, p+t-1]$, then $\Sigma'_{\geq p}(G, \alpha)$ denotes the sum of the colors of the edges of G. For a graph G and $p \in \mathbf{N}$, define the shifted edge-chromatic sum $\Sigma'_{\geq p}(G)$ as follows: $\Sigma'_{\geq p}(G) = \min_{\alpha} \Sigma'_{\geq p}(G, \alpha)$, where minimum is taken among all possible proper edge-colorings of G with colors $p, p+1, \ldots$. The theorem we are going to prove will be used in Section 5.

Theorem 5. *For any $n, p \in \mathbf{N}$, we have*

$$\Sigma'_{\geq p}(K_n) = \begin{cases} \dfrac{n(n-1)(2p+n-1)}{4}, & \text{if } n \text{ is odd}, \\ \dfrac{n(n-1)(2p+n-2)}{4}, & \text{if } n \text{ is even}. \end{cases}$$

Proof. Since for any r-regular graph G with n vertices, $\Sigma'(G) = \frac{nr(r+1)}{4}$ if and only if $\chi'(G) = r$ and, by Theorems 3 and 4, we obtain $\Sigma'_{\geq p}(K_n) = \frac{n(p+p+1+\cdots+p+n-2)}{2} = \frac{n(n-1)(2p+n-2)}{4}$ if n is even.

Now let n be an odd number and $n \geq 3$. In this case by Theorems 3 and 4, we have $s'(K_n) = \chi'(K_n) = n$. It is easy to see that in any proper n-coloring of K_n the missing colors at n vertices are all distinct. Hence,

$$\Sigma'_{\geq p}(K_n) = \frac{\frac{n^2(2p+n-1)}{2} - \frac{n(2p+n-1)}{2}}{2} = \frac{n(n-1)(2p+n-1)}{4}. \quad \square$$

Corollary 6. *For any $n \in \mathbf{N}$, we have*

$$\Sigma'(K_n) = \begin{cases} \dfrac{n(n^2-1)}{4}, & \text{if } n \text{ is odd}, \\ \dfrac{(n-1)n^2}{4}, & \text{if } n \text{ is even}. \end{cases}$$