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A Boolean theory of signatures for tonal scales

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We dedicate this paper to the memory of our co-author and friend Bruno Simeone, who sadly passed away while working on this manuscript. He introduced the new idea of Boolean signatures and wrote most of this paper

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ABSTRACT

We explore the concept of tonal signatures developed and put into musical practice by one of us (Mezzadri). A tonal signature of a scale S is a minimal subset of notes within S that is not contained in any scale S' different from S. We present a set covering model to find a smallest signature. We also show that the signatures of a scale are the prime implicants of a suitable monotone Boolean function represented by a Conjunctive Normal Form. On this ground, we introduce a more general notion of Boolean signature, depending on a Boolean operator. The computational machinery for generating Boolean signatures remains essentially the same. The richness and variety of Boolean signatures has a great potential for the development of new paradigms in polytonal harmony.

1. Introduction

In this paper we present an application of discrete mathematics to musical composition by means of tonal and Boolean signatures.

The tonal system – the way the intervals between notes of a scale are organized – has always been playing a strong role in western classical and improvised music. Western classical music has gradually evolved from the use of different modes up to the tonal system with specific hierarchical pitch relationships based on a main key called tonic. In tonal system, two modes have remained: the major and the minor. Here, we are interested in the tonal system and are considering the major mode and the minor and harmonic minor modes. This is an esthetic choice from which we name our research "tonal signature". Tonal harmony is well described in theory and practice in the book "Tonal harmony in concept and practice" by Forte [6]. The concept of signatures dealt with in this paper could be used for different set of scales, but the important thing about tonal signatures is that it refers to a tonal harmonic and melodic context which is part of our western musical background.

The purpose of tonal signatures is to find a minimal set of notes that can still be relevant of a scale, of its flavor or color. Since the writing of western polyphonic music, musicians and composers have been looking for systems and abstract

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Table 1
Characteristic vectors of the three scales in C.

Note	C Major	C minor	C harmonic
C	1	1	1
C♯	0	0	0
D	1	1	1
E♭	0	1	1
E	1	0	0
F	1	1	1
F♯	0	0	0
G	1	1	1
A♭	0	0	1
A	1	1	0
B♭	0	0	0
B	1	1	1

thinking that could first formalize their work then later provide generative ways to develop their music, with composition and later with improvisation. To be able to reduce the amount of information – like abstraction – makes the insight of the music clearer and easier to manipulate, to play with, to understand, it enables a faster thinking while playing and improvising with this concept. To a certain extent, even in some more experimental musical context, it is often desired to use extended scales in order to keep up a notion of center or *tonality*. Depending on the playing, on the speed and on the compositions, different degrees of complexity give a more or less clear readability, or intelligibility, of the tonal center. If it moves, we call it a tonal modulation. Mezzadri, a French flute player, has been looking for a minimal principle to deal with the readability of this center, i.e. a minimal set of notes which allows the listener to guess the tonal center. Should not a principle of minimalism also imply simplicity? The tonal signatures concept encompasses ideas that Mezzadri was initially expressing and experimenting intuitively with other musicians in his compositions, before coming to a formalization of it [11].

Our purpose, unlike the book "A Generative Theory of Tonal Music" by Lerdahl and Jackendoff [9] – even if it might share some common ideas – is not to model music understanding from the perspective of cognitive science, it is rather coming from the intuition of Mezzadri in order to formalize his ideas and to experiment them within real improvisation and composition contexts. We can refer to more generative music modeling and practice with the text "Generative music" edited by Collins and Brown [4]; an interesting and general overview on computer aided composition can be found in the two volumes of the "OM Composer's Book" edited by Agon, Assayag and Bresson [2,3].

Some music built on the concept of tonal signature has already been recorded on several CDs [12].

In this paper, we first introduce the tonal signature of a scale S as a minimal subset of notes within S that is not contained in any scale S' different from S. Then we formulate a set covering problem for finding a smallest signature and show that the signatures of a scale are the prime implicants of a suitable monotone Boolean function represented by a Conjunctive Normal Form (CNF). On this ground, we introduce the more general notion of Boolean signature, which depends not only on a scale, but also on a Boolean operator. From all the 16 possible binary Boolean operators, we come up with 14 groups of Boolean signatures, each one implying a different musical use and meaning. The richness and variety of Boolean signatures has a great potential for the development of new paradigms in polytonal harmony.

2. Basic definitions and notation

Let us denote by $N = \{C, C\sharp, D, E\flat, E, F, F\sharp, G, A\flat, A, B\flat, B\}$ the set of the 12 notes. Every subset of notes Q can be represented by a binary vector with 12 components, the *characteristic vector* $\mathrm{ch}(Q)$ of Q, defined as follows:

$$\mathrm{ch}_j(Q) = \begin{cases} 1, & \text{if } j \in Q \\ 0, & \text{if } j \notin Q \end{cases} \quad j = 1, \dots, 12.$$

For the sake of simplicity, a subset of N will be also denoted with the sequence of notes – ordered in the stated order – belonging to it.

A scale S is a nonempty subset of N. The most important scales in the western tradition are the Major (M), the minor (m) and the (minor) harmonic (h) scales, all having $|S| = 7$ on the same base set of 12 notes N. The characteristic vectors of the three scales C Major (C M), C minor (C m) and C (minor) harmonic (C h) are displayed in Table 1. Those of the other Major, minor, harmonic scales may be obtained from the characteristic vectors of the C M, C m, C h scales, respectively, by transposition i.e. by cyclic permutations of their components. Thus, all in all, there are 36 scales.

The scale-note matrix is the 36×12 binary matrix $A = [a_{ij}]$, whose columns correspond to the 12 notes (in the above order) and whose rows correspond to the 36 scales, in the order C, C♯, …, B♭, B; C m, C♯ m, …, B♭ m, B m; C h, C♯ h, …, B♭ h, B h. Let S_i be the ith scale in this order. By definition, for each $i = 1, \dots, 36$ and $j = 1, \dots, 12$,

$$a_{ij} = \begin{cases} 1, & \text{if } j \in S_i \\ 0, & \text{if } j \notin S_i. \end{cases} \tag{1}$$