



## Popular ranking

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### ABSTRACT

In this paper we take a new approach to the very old problem of aggregating preferences of multiple agents. We define the notion of *popular ranking*: a ranking of a set of elements is popular if there exists no other permutation of the elements that a majority of the voters prefer. We show that such a permutation is unlikely to exist: we show that a necessary but not sufficient condition for the existence of a popular ranking is Condorcet's paradox not occurring. In addition, we show that if Condorcet's paradox does not occur, then we can efficiently compute a permutation, which may or may not be popular, but for which the voters will have to solve an NP-hard problem to compute a permutation that a majority of them prefer.

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### 1. Introduction

How do you aggregate the preferences of multiple agents in a fair manner? This is a question that has occupied researchers for several centuries. Suppose we have  $k$  voters who each give their preferences on  $n$  candidates. How should the candidates be ranked to best represent the input? Condorcet [5] showed that there may not exist a “winner”: a candidate who beats all other candidates in a pairwise majority vote. Borda [4] and Condorcet [5] (and many others after them) proposed different ways of aggregating the preferences of the voters, and argued over which method is the right one. Only in the middle of the 20th century, Arrow [2] showed that there is no right method: there exists no aggregation method that simultaneously satisfies three natural criteria (non-dictatorship, independence of irrelevant alternatives and Pareto efficiency).

This negative result notwithstanding, we still want to find aggregate rankings based on voters' inputs. In this paper, we consider the case when, rather than selecting a winner, we would like to find a permutation of the candidates that represents the voters' inputs. Each voter's input is assumed to be a permutation of the candidates, where a candidate is ranked above another candidate, if the voter prefers the former to the latter candidate.

A common approach to this problem is to try and find a permutation that minimizes the sum of the distances to the voters' permutations, where in principle any distance(-like) function on permutations can be used, e.g. Kendall distance or Footrule distance. Young and Levenglick [10] show that the Kendall distance is the unique distance function such that the permutation(s) that minimize it have three desirable properties of being neutral, consistent and Condorcet. The Condorcet property means that, if there exists a permutation such that the order of every pair of elements is the order preferred by a majority, then this permutation has minimum distance to the voters' permutations. This distance was already proposed by Kemeny [8] for other reasons ([8] defines axioms on the *distance function*, and finds that the Kendall distance adheres to the axioms), and the problem of finding an optimal ranking with respect to this criterion is now known as Kemeny Rank Aggregation.

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In this paper, we suggest a new way of thinking about this problem. Suppose instead of minimizing the total distance from the voters' inputs, we want to find a permutation that makes a majority of the voters "happy"? Of course, a voter is happy when we follow her opinion exactly, and we cannot do this simultaneously for a majority of the voters, unless a majority of the voters is in total agreement. Therefore, our goal is to find a permutation such that there exists no other permutation that a majority of the voters prefer, in the sense that their Kendall distance to the alternative permutation is smaller. We call such a permutation a *popular ranking*.

Unfortunately, we show that such a permutation is unlikely to exist: it can only exist if Condorcet's paradox does not occur. Even worse than this, we show that if Condorcet's paradox does not occur, then it may still be the case that no popular ranking exists. The only positive news in this context is, perhaps paradoxically, an NP-hardness result: we show that if Condorcet's paradox does not occur, then we can efficiently compute a permutation, which may or may not be popular, but for which the voters will have to solve an NP-hard problem to compute a permutation that a majority of them prefer.

## Related work

Our work is inspired by Abraham et al. [1] where the notion of *popular matchings* is introduced. Popular ranking is also related to the problem of designing a voting mechanism in which the voters do not have an incentive to lie about their preferences. However, rather than considering deviations of a single voter, a popular solution is robust against deviations of a majority of the voters. We show that, if the input does not contain Condorcet's paradox, then there is a solution that may or may not be popular, but for which it is computationally hard for a majority of the voters to manipulate the output to their advantage. This result has a similar flavor as a result by Bartholdi et al. [3], who demonstrate a voting rule for deciding the "winner" of an election, for which it is computationally hard for a single voter to manipulate the output. We note that Bartholdi et al. [3] inspired many papers in recent years on the impossibility of manipulating elections under complexity assumptions; we refer the interested reader to the survey article by Faliszewski et al. [7].

## 2. Popular ranking

We are given a set of alternatives  $[n]$  (where the notation  $[n]$  means  $\{1, 2, \dots, n\}$ ) and a set of voters  $[k]$ , where each voter  $\ell$  has a complete ordering of the alternatives. We will denote these complete orderings of a voter  $\ell$  as a list of the alternatives, where an alternative earlier in the list is preferred to elements that succeed it, and use the notation  $\pi_\ell^{-1} : [n] \rightarrow [n]$  (the use of " $-1$ " will become clear shortly), where  $\pi_\ell^{-1}(i)$  is the alternative at position  $i$  in the ordering of voter  $\ell$ . Note that we can interpret  $\pi_\ell^{-1}$  as a permutation. Further, the inverse of  $\pi_\ell^{-1}$ , which we will denote by  $\pi_\ell$ , is well defined and can be interpreted as the position of the alternatives in the list of voter  $\ell$ . We will use list  $(\pi_\ell)$  to denote the ordered sequence  $(\pi_\ell^{-1}(1), \pi_\ell^{-1}(2), \dots, \pi_\ell^{-1}(n))$ .

The Kendall distance between two permutations  $\pi, \sigma$ , denoted by  $K(\pi, \sigma)$ , is defined as the number of pairwise disagreements of  $\pi$  and  $\sigma$ , i.e.  $K(\pi, \sigma) = \#\{i, j : \pi(i) < \pi(j) \text{ and } \sigma(i) > \sigma(j)\} + \#\{i, j : \pi(i) > \pi(j) \text{ and } \sigma(i) < \sigma(j)\}$ .

**Definition 1.** We say a permutation  $\pi$  is *popular*, if there is no  $\pi'$  and  $M \subseteq [k]$ , such that  $K(\pi_\ell, \pi') < K(\pi_\ell, \pi)$  for  $\ell \in M$ , and  $M$  is a strict majority of the voters, i.e.  $|M| > k/2$ .

Let  $S(i, j)$  be the subset of voters that prefer alternative  $i$  to  $j$ , i.e.  $S(i, j) = \{\ell \in [k] : \pi_\ell(i) < \pi_\ell(j)\}$ . We define the *majority graph*  $G = (V, A)$  as the directed graph which has a vertex for every  $i \in [n]$  and an arc  $(i, j)$  if a majority of the voters prefer  $i$  to  $j$ , i.e.  $|S(i, j)| > |S(j, i)|$ . Condorcet observed that such a graph may have a cycle; this is known as "Condorcet's paradox".

**Lemma 2.** No popular ranking exists if the majority graph has a directed cycle.

**Proof.** Given a majority graph with a directed cycle, we will show that for any ranking  $\pi$ , we can construct a ranking  $\pi'$  and give a subset of voters  $M$  where  $|M| > k/2$ , so that  $\pi'$  is preferred to  $\pi$  by all voters in  $M$ .

If we order the elements from left to right according to ranking  $\pi$ , then there must be some arc  $(i, j)$  in the majority graph that is a *back arc*, i.e. for which  $\pi(j) < \pi(i)$ . Let  $\pi'$  be the permutation we obtain by swapping  $i$  and  $j$ , i.e.  $\pi'(i) = \pi(j)$ ,  $\pi'(j) = \pi(i)$  and  $\pi'(t) = \pi(t)$  for all  $t \neq i, j$ . We will show that  $M = S(i, j)$  is a strict majority of the voters who prefer  $\pi'$  to  $\pi$ .

Call  $t$  an intermediary vertex if  $\pi(j) < \pi(t) < \pi(i)$ . When changing  $\pi$  into  $\pi'$ , we change the relative order of  $\{i, j\}$  and of  $\{i, t\}$ ,  $\{t, j\}$  for every intermediary vertex  $t$ . Now, for any voter  $\ell$  for whom  $\pi_\ell(i) < \pi_\ell(j)$ , and an intermediary vertex  $t$ , it must either be the case that  $\pi_\ell(i) < \pi_\ell(t)$  or  $\pi_\ell(t) < \pi_\ell(j)$  (or both). Hence the contribution of the two pairs  $\{i, t\}$  and  $\{t, j\}$  to  $K(\pi_\ell, \pi')$  is either strictly less than or equal to their contribution to  $K(\pi_\ell, \pi)$ . Finally, the contribution of pair  $\{i, j\}$  to  $K(\pi_\ell, \pi')$  is zero, and to  $K(\pi_\ell, \pi)$  it is one.

Hence  $K(\pi_\ell, \pi') < K(\pi_\ell, \pi)$  for each voter  $\ell \in M = S(i, j)$ , and this is a strict majority of the voters by the definition of the majority graph.  $\square$

If the majority graph is acyclic, then a popular ranking could exist. We consider the case when the majority graph is a tournament, i.e. for every  $i, j$  exactly one of the arcs  $(i, j)$  and  $(j, i)$  is in  $G$ . Note that the majority graph is always a tournament if the number of voters is odd. From the proof of Lemma 2, we know that the only permutation that *could* be popular is the permutation we obtain by topologically sorting the majority tournament. However, it is not the case that this ranking is always a popular ranking.

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