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Clique-perfectness of complements of line graphs

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ABSTRACT

A graph is clique-perfect if the maximum number of pairwise disjoint maximal cliques equals the minimum number of vertices intersecting all maximal cliques for each induced subgraph. In this work, we give necessary and sufficient conditions for the complement of a line graph to be clique-perfect and an $O(n^2)$-time algorithm to recognize such graphs. These results follow from a characterization and a linear-time recognition algorithm for matching-perfect graphs, which we introduce as graphs where the maximum number of pairwise edge-disjoint maximal matchings equals the minimum number of edges intersecting all maximal matchings for each subgraph. Thereby, we completely describe the linear and circular structure of the graphs containing no bipartite claw, from which we derive a structure theorem for all those graphs containing no bipartite claw that are Class 2 with respect to edge-coloring.

1. Introduction

Numerous major theorems in combinatorics are formulated in terms of min–max relations of dual graph parameters.

Perfect graphs were defined by Berge in terms of a min–max inequality involving clique and chromatic number. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors needed to assign different colors to adjacent vertices of G. The maximum size of a clique in G is its *clique number* $\omega(G)$. Clearly, the min–max type inequality $\omega(G) \leq \chi(G)$ holds for every graph G. Berge [3] called a graph G *perfect* if and only if the equality $\omega(H) = \chi(H)$ holds for each induced subgraph H of G.

An important result about perfect graphs is the *Perfect Graph Theorem* which states that the complement of a perfect graph is also perfect [29,40]. Thus, a graph G is perfect if and only if clique and chromatic number coincide for each induced subgraph of its complement $\overline{G}$. The clique number of $\overline{G}$ is the *stability number* $\alpha(G)$, which is the maximum number of pairwise nonadjacent vertices of G. The chromatic number of $\overline{G}$ is the *clique covering number* $\theta(G)$, which is the minimum number of cliques of G covering all its vertices. Hence, the min–max type inequality $\alpha(G) \leq \theta(G)$ holds for every graph G and, by virtue of the Perfect Graph Theorem, a graph G is perfect if and only if $\alpha(H) = \theta(H)$ holds for each induced subgraph H of G.

A *hole* or *antihole* in a graph G is an induced subgraph isomorphic to the chordless cycle on k vertices C_k or its complement $\overline{C_k}$, respectively, for some $k \geq 5$. If k is odd, then the hole or antihole is *odd*; otherwise it is *even*. Berge [3] conjectured that a graph is perfect if and only if it has no odd holes and no odd antiholes. This conjecture was proved to be true about 40 years later and is now known as the *Strong Perfect Graph Theorem* [19].

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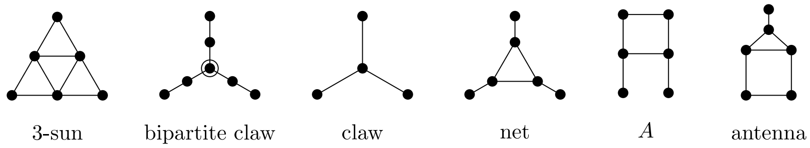

Fig. 1. Some small graphs. The circled vertex is the *center* of the bipartite claw.

Theorem 1.1 (*Strong Perfect Graph Theorem [19]*). *A graph is perfect if and only if it has no odd holes and no odd antiholes.*

A polynomial-time recognition algorithm for perfect graphs was given in [18].

The class of clique-perfect graphs is defined by requiring equality in a min–max type inequality related to the Kőnig property of the family of maximal cliques. Consider a family $\mathcal{F}$ of nonempty subsets of a finite ground set X, then the *transversal number* $\tau(\mathcal{F})$ is the minimum number of elements of X needed to intersect every member of $\mathcal{F}$ and the *matching number* $\nu(\mathcal{F})$ of $\mathcal{F}$ is the maximum size of a collection of pairwise disjoint members of $\mathcal{F}$. If these two numbers coincide, the family $\mathcal{F}$ is said to have the *Kőnig property* [4].

Let $\mathcal{Q}$ be the family of all maximal cliques of G. A collection of pairwise disjoint maximal cliques of a graph is a *clique-independent set* and a vertex set intersecting every maximal clique of a graph is a *clique-transversal*. Accordingly, we call $\nu(\mathcal{Q})$ the *clique-independence number* $\alpha_c(G)$ and $\tau(\mathcal{Q})$ the *clique-transversal number* $\tau_c(G)$. Clearly, the min–max type inequality $\alpha_c(G) \leq \tau_c(G)$ holds for every graph G. A graph G is *clique-perfect* [30] if $\alpha_c(H) = \tau_c(H)$ holds for each induced subgraph H of G. In other words, a graph G is clique-perfect if and only if, for each induced subgraph of G, the family of all maximal cliques has the Kőnig property.

The Kőnig property has its origins in the study of matchings and transversals in bipartite graphs. The *matching number* $\nu(G)$ of a graph G is the maximum size of a *matching* (a set of vertex-disjoint edges) and the *transversal number* $\tau(G)$ is the minimum size of a *vertex cover* (a set of vertices intersecting every edge). Clearly, the min–max type inequality $\nu(G) \leq \tau(G)$ holds for every graph G. In 1931, Kőnig [36] and Egerváry [27] proved that every bipartite graph B satisfies $\nu(B) = \tau(B)$. This result is now known as the *Kőnig–Egerváry Theorem*. Notice that if B is bipartite, then $\alpha_c(B) = \nu(B) + i(B)$ and $\tau_c(B) = \tau(B) + i(B)$ where $i(B)$ denotes the number of isolated vertices of B; consequently, $\alpha_c(B) = \tau_c(B)$ if and only if $\nu(B) = \tau(B)$. Therefore, since each induced subgraph of a bipartite graph is also bipartite, the Kőnig–Egerváry Theorem can be restated by saying that every bipartite graph is clique-perfect. Apart from bipartite graphs, some other graph classes are known to be clique-perfect: comparability graphs [1], balanced graphs [5], complements of forests [7], and distance-hereditary graphs [37].

It is important to mention that not all clique-perfect graphs are perfect and that not all perfect graphs are clique-perfect. For instance, the even antihole $\overline{C_{6k+2}}$ is perfect but not clique-perfect, whereas the odd antihole $\overline{C_{6k+3}}$ is clique-perfect but not perfect, for each $k \geq 1$. In fact, we have:

Theorem 1.2 (*[26,30]*). *A hole C_n is clique-perfect if and only if n is even. An antihole $\overline{C_n}$ is clique-perfect if and only if n is a multiple of 3.*

Notice also that if the equality $\alpha_c(G) = \tau_c(G)$ holds for a graph G, then the same equality may not hold for all its induced subgraphs. For instance, every graph G in the class of *dually chordal graphs* [14] satisfies the equality $\alpha_c(G) = \tau_c(G)$, but dually chordal graphs are not clique-perfect in general; e.g., the 5-*wheel* (the graph that arises from C_5 by adding a vertex adjacent to every other vertex) is dually chordal but it is not clique-perfect because it contains an induced C_5, for which $\alpha_c(C_5) = 2$ but $\tau_c(C_5) = 3$.

Unlike perfect graphs, the class of clique-perfect graphs is not closed under graph complementation; e.g., the net and the 3-sun (see Fig. 1) are complement graphs of each other such that the former is clique-perfect but the latter is not clique-perfect. Moreover, a complete characterization of clique-perfect graphs by forbidden induced subgraphs is not known either. Another open question regarding clique-perfect graphs is the computational complexity of their recognition problem. Nevertheless, some partial results in this direction appeared in [8,9,11,38], where necessary and sufficient conditions for a graph G to be clique-perfect in terms of forbidden induced subgraphs as well as polynomial-time algorithms for deciding whether a given graph G is clique-perfect were found when restricting G to belong to one of several different graph classes. Interestingly, the problems of determining $\alpha_c(G)$ and $\tau_c(G)$ are both NP-hard even if G is a split graph [17] and determining $\tau_c(G)$ is NP-hard even if G is a triangle-free graph [28]. More NP-hardness results of this type for α_c and τ_c were proved in [30]. Some polynomial-time algorithms for determining $\alpha_c(G)$ and $\tau_c(G)$ when G belongs to one of several different graph classes were devised in [1,13,17,22,24–26,30,37].

The *line graph* $L(H)$ of a graph H is the graph whose vertices are the edges of H and such that, for every two different edges e and f of H, ef is an edge of $L(H)$ if and only if e and f share an endpoint. A graph G is a *line graph* [51] if it is the line graph of some graph H; if so, H is called a *root graph* of G. Perfectness of line graphs (or, equivalently, of their complements) was studied in [42,43]. In [8], clique-perfectness of line graphs was characterized by forbidden induced subgraphs, as follows (see Fig. 1 for a 3-sun).

Theorem 1.3 (*[8]*). *If G is a line graph, then G is clique-perfect if and only if G contains no induced 3-sun and has no odd hole.*